

The basins of attraction for optimal fourth-eighth order methods

S. Parimala¹

¹Department of Mathematics, Saradha Gangadharan College, Velrampet, Puducherry-605004, India

*Corresponding author: parimalas@pec.edu

Article Info

Keywords: Newton's method, Basins of attraction, Non-linear equation.

Received: 05.07.2025;

Accepted: 20.09.2025;

Published: 13.10.2025

Abstract

We developed a completely new, eighth- and fourth-order optimal approach for solving nonlinear equations in this study. With three function evaluations, the new methods' convergence order is four; with four function evaluations, it is eight. Furthermore, according to the Kung-Traub hypothesis, it is optimal. In order to illustrate their dynamic behaviour on the complex plane, the basins of attraction are provided.



© 2024 by the author's. The terms and conditions of the Creative Commons Attribution (CC BY) license apply to this open access article.

1. Introduction

The nonlinear equation $f(x) = 0$ is one of the most significant problems in engineering, scientific computing, and applied mathematics in general. The method that is most frequently used to solve nonlinear equations is Newton's iteration method. Newton's approach has been refined by other academics to achieve higher order convergence and more accurate findings, see example [1–7]. In addition, the efficiency index (EI) is a widely used technique to evaluate the effectiveness of various iterative approaches. The definition of this index is $p^{1/m}$, where m is the number of functional evaluations required at each iteration and p is the convergence order. If and only if the iterative method with m functional evaluations has an order of convergence equal to 2^{m-1} , according to the conjecture of [8]. The most effective iterative methods for varied convergence orders have been developed by numerous scholars. Typically, the composition methodology is used to construct an optimal method, along with a few interpolations and approximations to minimise the amount of functional evaluations needed at each iteration. Various optimal fourth order and eighth order iterative techniques were developed, see for example [1, 9–12]. Further, we studied the behaviour of iterative scheme in the complex plane. Furthermore, a number of researchers have applied these concepts to many iterative schemes [1–6, 13–15], which discussed the basin of attraction of a few well-known iterative schemes.

The rest of the paper is set up as follows. The proposed strategies have been developed and their convergence analysis is covered in Section 2. Section 3 uses basins of attraction to study the suggested methods in the complex plane. Section 4 provides concluding observations.

2. Construction of proposed methods

We will define an Iterative Function (I.F.) by $x_{n+1} = \psi(x)$. Using the additional information at x , $\phi_1(x), \dots, \phi_i(x)$, $i \geq 1$, let x_{n+1} be calculated. Nothing from the past is utilised. Consequently,

$$x_{n+1} = \psi(x, \phi_1(x), \dots, \phi_i(x)). \quad (1)$$

A multipoint I.F. without memory is then defined as ψ .

The Newton-Raphson (also known as Newton-I.F.) ($2^{nd}NR$) is provided by

$$\psi_{2^{nd}NR}(x) = x - u(x), u(x) = \frac{f(x)}{f'(x)}. \quad (2)$$

With two function evaluations, the $2^{nd}NR$ I.F. is a one-point I.F. that meets the Kung-Traub conjecture for $d = 2$. Also, $EL_{2^{nd}NR} = 1.414$.

2.1. Proposed optimal fourth order I.F

In this way, we attempt to derive a new optimal fourth order I.F.

$$\begin{aligned} \psi_{4^{th}PM}(x) &= \psi_{2^{nd}NR}(x) - H(\tau) \frac{f(\psi_{2^{nd}NR}(x))}{f'(x)}. \\ H(\tau) &= H(1) + (\tau - 1) H'(1) + (\tau - 1)^2 H''(1) + \dots \text{ and } \tau = 1 - \frac{f(\psi_{2^{nd}NR}(x))}{f(x)}. \end{aligned} \quad (3)$$

In the following Theorem we discuss the choice of the parameter $|H''(1)|$ for which the proposed method (3) has the optimal fourth order convergence.

Assume that the function $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$ has continuous derivatives and is suitably smooth. If x_0 is selected in a suitably small neighborhood of x^* and $f(x)$ has a simple root x^* in the open interval D , then the approach (3) has fourth order convergence, when

$$H(1) = 1, H'(1) = -2, |H''(1)| < \infty \quad (4)$$

The error equation is satisfied.

$$e_{n+1} = \left(\left(5 - \frac{H''(1)}{2} \right) c_2^3 - c_2 c_3 \right) e^4 + O(e^5). \quad (5)$$

$$c_j = \frac{f^{(j)}(x^*)}{j! f'(x^*)}, j = 2, 3, 4, \dots \text{ and } e = x - x^*.$$

Proof. Let $\tilde{e} = \psi_{2^{nd}NR}(x) - x^*$, $\hat{e} = \psi_{4^{th}PM}(x) - x^*$. Extending $f(x)$ and $f'(x)$ Around x^* using Taylor's technique, we have

$$f(x) = f'(x^*) \left(e + c_2 e^2 + c_3 e^3 + c_4 e^4 + c_5 e^5 + c_6 e^6 + c_7 e^7 + c_8 e^8 + O(e^9) \right) \quad (6)$$

and

$$f'(x) = f'(x^*) \left(1 + 2c_2 e + 3c_3 e^2 + 4c_4 e^3 + 5c_5 e^4 + 6c_6 e^5 + 7c_7 e^6 + 8c_8 e^7 + 9c_9 e^8 + O(e^9) \right) \quad (7)$$

Thus,

$$\begin{aligned} \tilde{e} &= c_2 e^2 + \left(2c_3 - 2c_2^2 \right) e^3 + \left(-7c_2 c_3 + 4c_2^3 + 3c_4 \right) e^4 + \left(-8c_2^4 \right. \\ &\quad \left. + 20c_2^2 c_3 - 6c_3^2 - 10c_2 c_4 + 4c_5 \right) e^5 + \left(16c_2^5 - 52c_2^3 c_3 + 28c_2^2 c_4 - 17c_3 c_4 \right. \\ &\quad \left. + c_2(33c_3^2 - 13c_5) + 5c_6 \right) e^6 - 2 \left(16c_2^6 - 64c_2^4 c_3 - 9c_3^2 + 36c_2^3 c_4 + 6c_4^2 + 9c_2^2(7c_3^2 \right. \\ &\quad \left. - 2c_5) + 11c_3 c_5 + c_2(-46c_3 c_4 + 8c_6) - 3c_7 \right) e^7 + \left(64c_2^7 - 304c_2^5 c_3 \right. \\ &\quad \left. + 176c_2^4 c_4 + 75c_2^3 c_4 + c_2^2(408c_3^2 - 92c_5) - 31c_4 c_5 - 27c_3 c_6 \right. \\ &\quad \left. + c_2^2(-348c_3 c_4 + 44c_6) + c_2(-135c_3^2 + 64c_4^2 + 118c_3 c_5 - 19c_7) + 7c_8 \right) e^8 + \dots \end{aligned} \quad (8)$$

Using Taylor's approach, we may expand $f(\psi_{2^{nd}NR}(x))$ about x^* and obtain

$$f(\psi_{2^{nd}NR}(x)) = f'(x^*) \left(\tilde{e} + c_2 \tilde{e}^2 + c_3 \tilde{e}^3 + c_4 \tilde{e}^4 + O(\tilde{e}^5) \right) \quad (9)$$

We obtain by simplifying and substituting these equations (6)-(8) and (4) in the (3).

$$\psi_{4^{th}PM}(x) - x^* = \left(\left(5 - \frac{H''(0)}{2} \right) c_2^3 - c_2 c_3 \right) e^4 + O(e^5).$$

This shows that fourth-order convergence is achieved by the suggested classes of approaches. $\square$

We are able to generate a new optimal fourth order method in (4) by selecting any random value for $H''(1)$. Selecting $H''(1) = 2$ yields new suggested approaches as follows.

$$\begin{aligned} \psi_{4^{th}PM}(x) &= \psi_{2^{nd}NR}(x) - H(\tau) \frac{f(\psi_{2^{nd}NR}(x))}{f'(x)}. \\ H(\tau) &= 1 - 2(\tau - 1) + (\tau - 1)^2 \text{ and } \tau = 1 - \frac{f(\psi_{2^{nd}NR}(x))}{f(x)}. \end{aligned} \quad (10)$$

This method (10) has the following error equation $\psi_{4^{th}PM}(x) - x^* = (3c_2^3 - c_2 c_3) e^4 + O(e^5)$. $EL_{4^{th}PM} = 1.587$ is the efficiency of the method (15).

2.2. An eighth-order optimum technique

Next, we try the following method to obtain a new optimal eighth order I.F.

$$\psi_{8th PM}(x) = \psi_{4th PM}(x) - \frac{f(\psi_{4th PM}(x))}{f'(\psi_{4th PM}(x))}.$$

With five function evaluations, the aforementioned one exhibits eighth order convergence. However, this is not the best approach. In order to estimate $f'(\psi_{4th PM}(x))$, we must minimise a function while maintaining the same convergence order. This polynomial is used to estimate the optimal.

$$q(t) = b_3(t-x)^3 + b_2(t-x)^2 + b_1(t-x) + b_0, \quad (11)$$

which fulfils

$$q'(x) = f'(x), q(x) = f(x), q(\psi_{2nd NR}(x)) = f(\psi_{2nd NR}(x)), q(\psi_{4th PM}(x)) = f(\psi_{4th PM}(x)).$$

When the aforementioned requirements are applied to (11), there are generated four linear equations: b_0, b_1, b_2 , and b_3 . $b_0 = f(x)$ and $b_1 = f'(x)$ follow from $q(x) = f(x)$, $q'(x) = f'(x)$. b_2 and b_3 are found by solving these equations:

$$\begin{aligned} f(\psi_{2nd NR}(x)) &= b_3(\psi_{2nd NR}(x) - x)^3 + b_2(\psi_{2nd NR}(x) - x)^2 + f'(x)(\psi_{2nd NR}(x) - x) + f(x) \\ f(\psi_{4th PM}(x)) &= b_3(\psi_{4th PM}(x) - x)^3 + b_2(\psi_{4th PM}(x) - x)^2 + f'(x)(\psi_{4th PM}(x) - x) + f(x). \end{aligned}$$

Therefore, by using divided differences, the aforementioned equations become simpler to

$$f[\psi_{2nd NR}(x), x, x] = b_2 + b_3(\psi_{2nd NR}(x) - x) \quad (12)$$

$$f[\psi_{4th PM}(x), x, x] = b_2 + b_3(\psi_{4th PM}(x) - x) \quad (13)$$

Equations (12) and (13) can be solved to yield

$$\begin{aligned} b_2 &= \frac{f[\psi_{2nd NR}(x), x, x](\psi_{4th PM}(x) - x) - f[\psi_{4th PM}(x), x, x](\psi_{2nd NR}(x) - x)}{\psi_{4th PM}(x) - \psi_{2nd NR}(x)}, \\ b_3 &= \frac{f[\psi_{4th PM}(x), x, x] - f[\psi_{2nd NR}(x), x, x]}{\psi_{4th PM}(x) - \psi_{2nd NR}(x)}. \end{aligned} \quad (14)$$

Furthermore, we have the estimation using eq. (14).

$$f'(\psi_{4th PM}(x)) \approx q'(\psi_{4th PM}(x)) = b_1 + 2b_2(\psi_{4th PM}(x) - x) + 3b_3(\psi_{4th PM}(x) - x)^2.$$

Lastly, we provide a fresh, eighth-order optimum technique as

$$\psi_{8th PM}(x) = \psi_{4th PM}(x) - \frac{f(\psi_{4th PM}(x))}{f'(x) + 2b_2(\psi_{4th PM}(x) - x) + 3b_3(\psi_{4th PM}(x) - x)^2}. \quad (15)$$

$El_{8thYM} = 1.682$ is the efficiency of the approach (15).

We use MATHEMATICA software to demonstrate the convergence analysis of the suggested I.F.s (15). Assume that the function $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$ is sufficiently smooth and has derivatives that are continuous. When x_0 is selected within a suitably small neighbourhood of x^* and $f(x)$ has a simple root x^* in the open interval D , the (15) is of eighth order convergence and fulfils the error equation:

$$\psi_{8th PM}(x) - x^* = c_2^2(4c_2^2 - c_3)(4c_2^3 - c_2c_3 + c_4)e^8 + O(e^9) \quad (16)$$

3. Basins of attraction

Analysing the rational function's dynamic behaviour in relation to an iterative process provides valuable insights into the method's stability and convergence. [14] and [16] provide fundamental definitions and dynamic notions of rational functions.

Applying our iterative methods, We pick a square with 256×256 points that is $\mathbb{R} \times \mathbb{R} = [-2, 2] \times [-2, 2]$. We start in every $z^{(0)}$ in the square. If, for a maximum of 100 iterations, the sequence generated by the iterative technique attempts a zero z_j^* of the polynomial with a tolerance $|f(z^{(k)})| < 1e-4$, we conclude that $z^{(0)}$ is in the basin of attraction of this zero. We label this point $z^{(0)}$ with colours if $|z^{(N)} - z_j^*| < 1e-4$.

This is done if the iterative technique, it begins in $z^{(0)}$ and, in N iterations ($N \leq 100$), reaches a zero. We determine that the starting point has diverged if $N > 50$, and we apply a dark blue colour. The following describes the basins of attraction for the Newton's method and a few higher order Newton-type methods for finding the complex roots of the polynomials $p_1(z) = z^3 - 1$ and $p_2(z) = z^5 - 1$.

Figure 1 displays the polynomiographs for the approaches to the polynomials $p_1(z)$ and $p_2(z)$ for the NR_2 . The polynomiographs for the fourth order iterative approaches for the polynomial $p_1(z)$ are displayed in Figure 2. The polynomiographs for the ninth order iterative approaches for the polynomial $p_1(z)$ are displayed in Figure 3. The polynomiographs for the fourth order iterative approaches for the polynomial $p_2(z)$ are displayed in Figure 4. The polynomiographs for the ninth order iterative approaches for the polynomial $p_2(z)$ are displayed in Figure 5.

It is noted that the performance of the approaches NR_2 , $4th PM$, and $8th PM$ is remarkable in the $p_1(z)$. In close proximity of the boundary points, the methods SB_4 , KT_8 , and LW_8 , exhibit some chaotic behaviour. In this scenario, the approaches CM_4 , SJ_4 , $PMPD_8$, SA_8 , and $CFGT_8$ are sensitive to the initial guess selection.

Also note that the approaches $4th PM$ and $8th PM$ exhibit some chaotic behaviour in the vicinity of the boundary points for $p_2(z)$. NR_2 , SB_4 , CM_4 , and SJ_4 are the techniques KT_8 . In this instance, the values of LW_8 , $PMPD_8$, SA_8 , and $CFGT_8$ are all sensitive to the initial guess made.

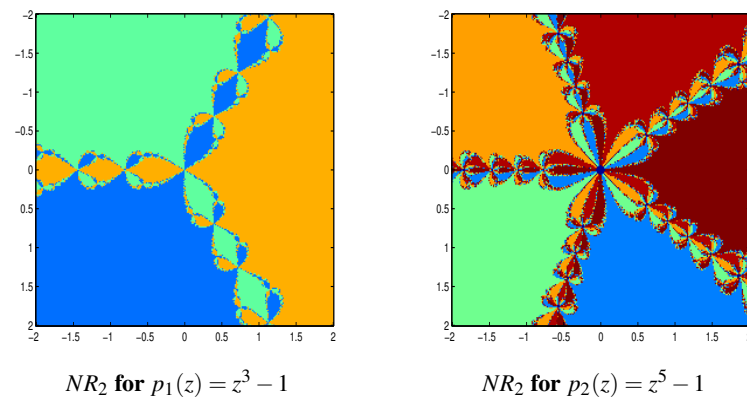


Figure 1: Newton's technique (NR_2) basins of attraction

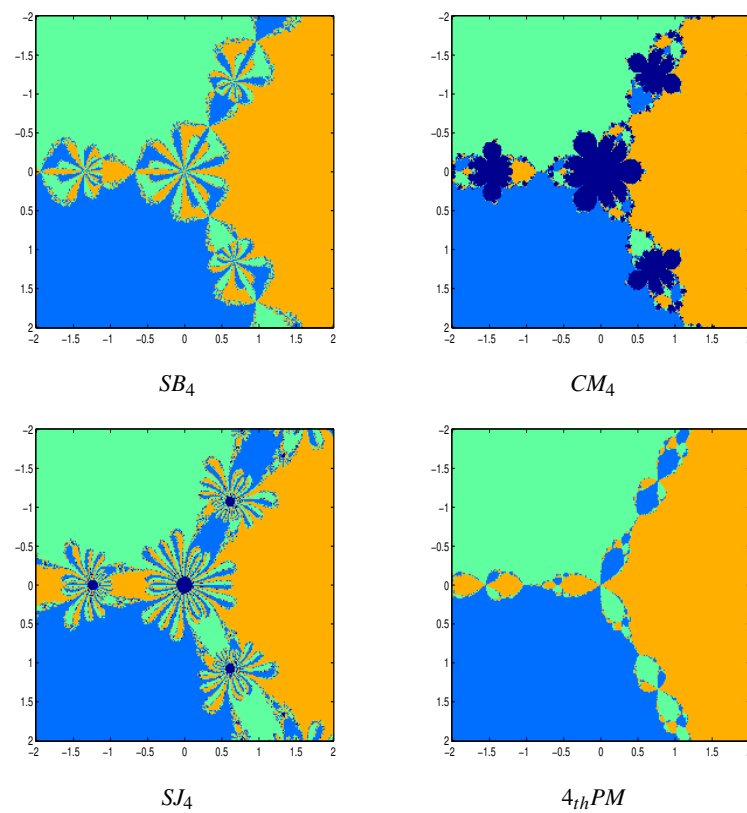


Figure 2: $p_1(z) = z^3 - 1$ basins of attraction

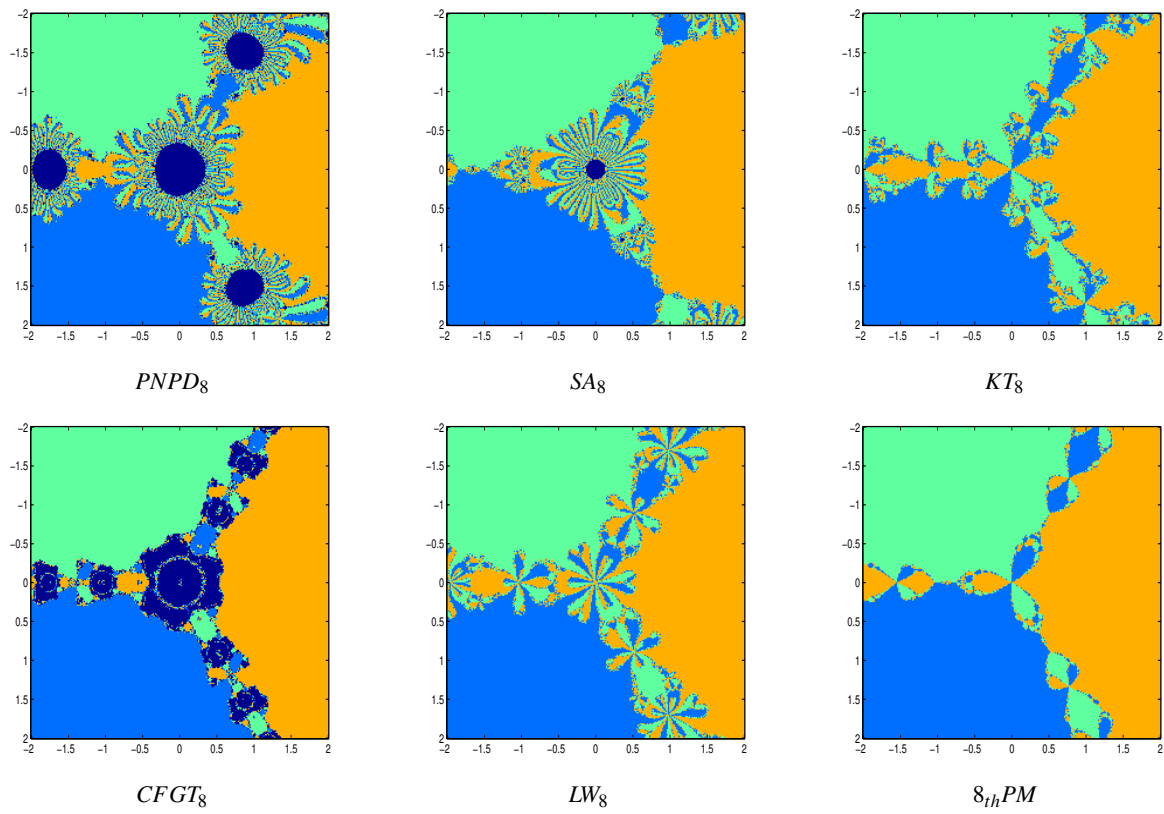


Figure 3: $p_1(z) = z^3 - 1$ basins of attraction

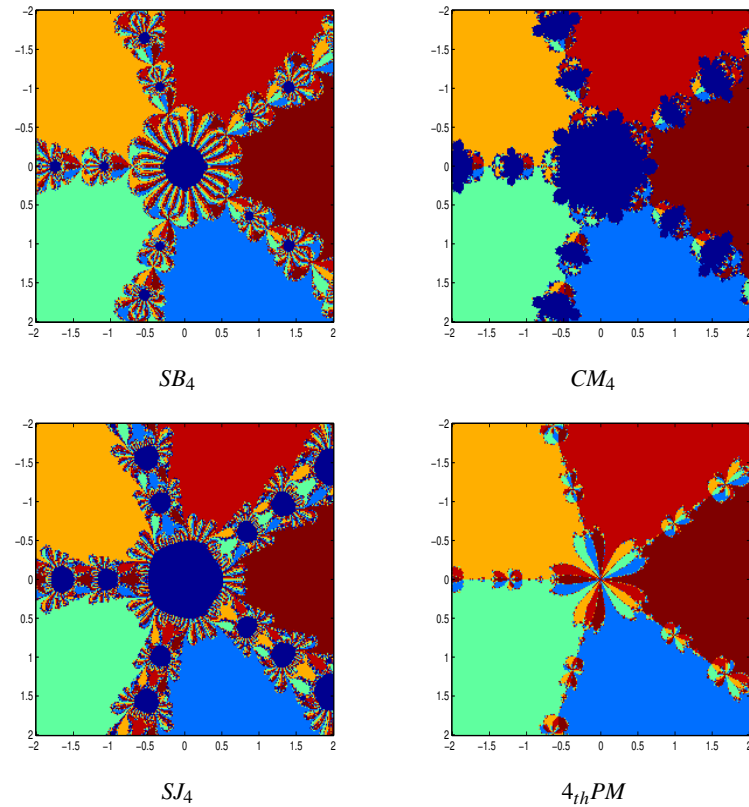


Figure 4: $p_1(z) = z^5 - 1$ basins of attraction

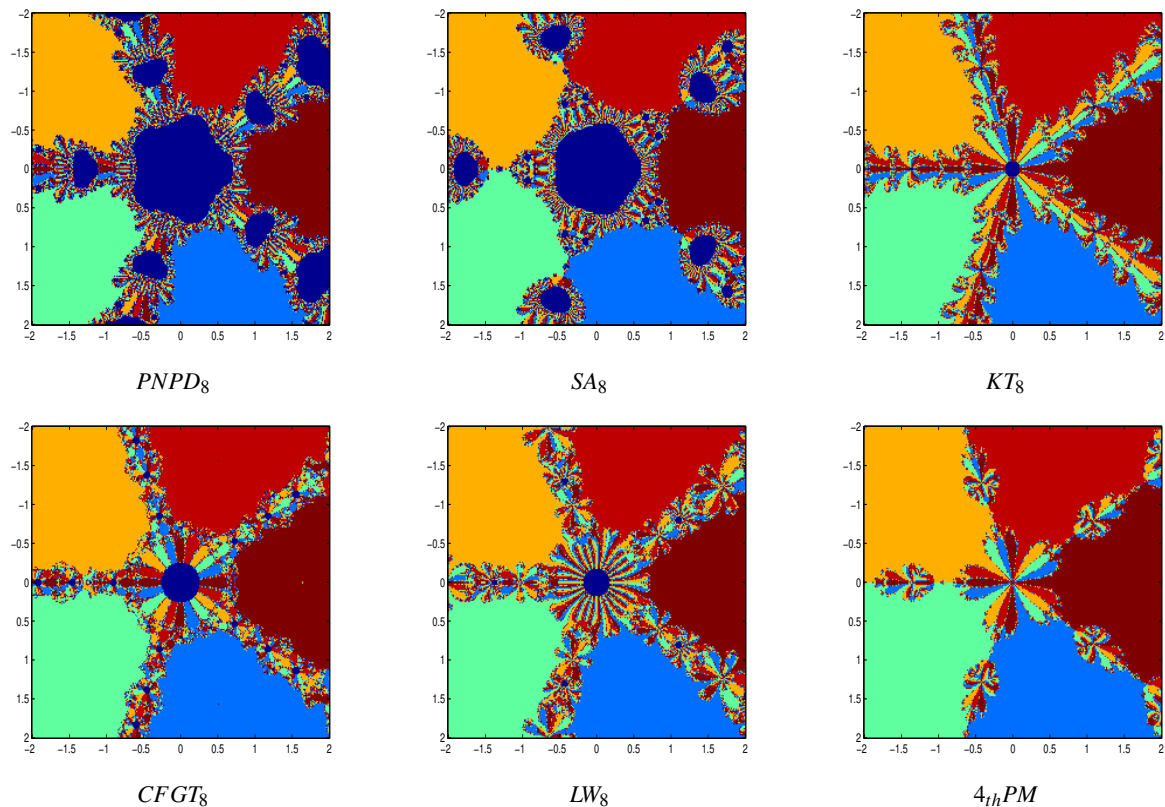


Figure 5: $p_1(z) = z^5 - 1$ basins of attraction.

4. Concluding Remarks

In this work, we established a family of iterative algorithms for solving nonlinear equations that is optimal at the fourth and eighth orders, respectively. Three and four function evaluations are needed for the approach to obtain an order of convergence of four and eight, respectively. The Kung-Traub conjecture is met in the sense of convergence analysis and numerical examples. To demonstrate the superiority of the proposed methods, we have tested a few examples with both the proposed schemes and certain recognised schemes. By displaying their corresponding fractals, more research has been done on the complex plane to uncover the basins of attraction of such approaches for solving nonlinear equations.

References

- [1] D. K. R. Babajee and K. Madhu. Comparing two techniques for developing higher order two-point iterative methods for solving quadratic equations. *SeMA Journal*, DOI: 10.1007/s40324-018-0174-0, 2018.
- [2] Alicia Cordero, Jose L. Hueso, Eulalia Martinez, and Juan R. Torregrosa. A family of iterative methods with sixth and seventh order convergence for nonlinear equations. *Mathematical and Computer Modelling*, 52:1490–1496, 2010.
- [3] J. H. Curry, L. Garnett, and D. Sullivan. On the iteration of a rational function: computer experiments with newton’s method. *Communications in Mathematical Physics*, 91:267–277, 1983.
- [4] F Soleymani, D K R Babajee, and M Sharifi. Modified Jarratt method without memory with twelfth-order convergence. *Annals of the University of Craiova, Mathematics and Computer Science Series*, 39:21–34, 2012.
- [5] E. R. Vrscay. Julia sets and mandelbrot-like sets associated with higher order schroder rational iteration functions: a computer assisted study. *Mathematics of Computation*, 46:151–169, 1986.
- [6] E. R. Vrscay and W. J. Gilbert. Extraneous fixed points, basin boundaries and chaotic dynamics for schroder and konig rational iteration functions. *Numerische Mathematik*, 52:1–16, 1987.
- [7] K. Madhu. New higher order iterative methods for solving nonlinear equations and their basins of attraction. *Current Research in Interdisciplinary Studies*, 2:1–15, 2023.
- [8] H.T. Kung and J.F. Traub. Optimal order of one-point and multi-point iteration. *J. Assoc. Comput. Math.*, 21:643–651, 1974.
- [9] S. Abdullah, N. Choubey, and S. Dara. Optimal fourth- and eighth-order iterative methods for solving nonlinear equations with basins of attraction. *J. Appl. Math. Comput.*, 70:3477–3507, 2024.

- [10] S. Abdullah, N. Choubey, and S. Dara. Two novel with and without memory multi-point iterative methods for solving non-linear equations. *Communications in Mathematics and Applications*, 15:9–31, 2024.
- [11] X. Wang and J. Li. Two step optimal jarratt-type fourth order methods using two weight functions for solving nonlinear equations. *Current Research in Interdisciplinary Studies*, 1:20–26, 2022.
- [12] X. Wang and J. Li. Higher order multi-point iterative methods for finding gps user position. *Current Research in Interdisciplinary Studies*, 1:27–35, 2022.
- [13] S Amat, S Busquier, and S Plaza. Dynamics of a family of third-order iterative methods that do not require using second derivatives. *Appl. Math. Comp.*, 154:735–746, 2004.
- [14] S. Amat, S. Busquier, and S. Plaza. Review of some iterative root-finding methods from a dynamical point of view. *SCIENTIA Series A: Mathematical Sciences*, 10:3–35, 2004.
- [15] Y. Tao and K. Madhu. Optimal fourth, eighth and sixteenth order methods by using divided difference techniques and their basins of attraction and its application. *Mathematics*, 7:322, 2019.
- [16] M. Scott, B. Neta, and C. Chun. Basin attractors for various methods. *Applied Mathematics and Computation*, 218:2584–2599, 2011.