Applied Mathematics on Science and Engineering, 3(1):16–26, 2026. DOI: 10.65036/amse312

Applied Mathematics on Science and Engineering

ISSN (Online): 3048-8362

Journal homepage: https://furthersci.org/journal-details.php?journal-id=49

Research Article

Fixed point approach for relation-theoretic contractive type mappings in $\mathscr{F}$-metric spaces with application to boundary value problems

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Article Info

Keywords: *Fixed point, $\mathscr{F}$-metric space, binary relation-theoretic contractive type mapping.*

MSC 2020: *54H25, 47H10.*

Received: *08.02.2026;*
Accepted: *05.06.2026;*
Published: *30.06.2026*

Abstract

The aim of this article is to introduce the notion of $\mathscr{F}$-metric spaces and establish some fixed point results for relation-theoretic contractive type mapping in a complete $\mathscr{F}$-metric space. These contributions extend the existing literature on relation-theoretic mapping and fixed point theory. Through illustrative numerical examples, we showcase the practical applicability of the theoretical findings. Also, we explore as an application, the solution for two points boundary value problems.

1. Introduction

Fixed point theory is among the most fascinating fields of research in the evolution of nonlinear analysis. One of the fundamental conclusions of fixed point theory, the Banach fixed point theorem [1], is crucial for establishing the existence and uniqueness of solutions to a wide range of mathematical issues. Building upon Banach's work, Turinici [2] extended the Banach fixed point theorem to the concept of order-theoretic fixed point results, which provided a novel approach to proving fixed point theorems. In this direction, Ran and Reurings [3] developed an order-theoretic version of the Banach contraction principle and demonstrated its application to matrix equations. Samet and Turinici [4] further advanced this line of research by establishing fixed point results for nonlinear contractions based on the symmetric closure of an amorphous binary relation. By extending the Banach contraction principle to arbitrary binary relations, Alam and Imdad [5] created a relation-theoretic fixed point theorem that unifies a number of previously known order-theoretic conclusions.

A new version of the Banach contraction principle for complete metric spaces with a binary relation was presented by Alam and Imdad [6]. In contrast to earlier fixed point theorems, their method relies on relation-theoretic analogues of contraction, completeness, and continuity rather than the conventional metrical definitions of these terms.

Moreover, Jleli and Samet [7] introduced a contemporary metric space, which is referred to as $\mathscr{F}$-metric space, to extend the classical metric space. In this direction, Alnaser et al. [8] used this concept and proved some fixed point results in $\mathscr{F}$-metric space with some open problems such as fixed point theorems for L-fuzzy mappings.

Recently, Tomar and Joshi [9] introduced the notion of $\mathscr{F}$-metric space as a generalization of traditional metric space and established some fixed point results in the context of relation-theoretic contractions.

Motivated by the significance of the generalizations, extensions and improvements of recent and classical results in literature to the framework of an $\mathscr{F}$-metric space, we introduce the concepts of $\mathscr{F}$-metric spaces and subsequently establish some fixed point results relation-theoretic contractive type mapping within the framework of complete $\mathscr{F}$-metric spaces. To bolster our findings, we offer illustrative numerical examples demonstrating the practical application of the presented results. Furthermore, we explore as an application, the solution for two points boundary value problems.

2. Preliminaries

We begin this section by presenting the concept of relation-theoretic contractive type mapping and $\mathscr{F}$-metric space.

Definition 2.1 [10] A binary relation R on a nonempty set X is called reflexive if $(x,x)\in R$ for all $x\in X$, symmetric if whenever $(x,y)\in R$ then $(y,x)\in R$, antisymmetric if whenever $(x,y)\in R$ and $(y,x)\in R$ then $x=y$, transitive if whenever $(x,y)\in R$ and $(y,z)\in R$ then $(x,z)\in R$, and complete or connected or dichotomous if $[x,y]\in R$ for all $x,y\in X$.

Definition 2.2 [11] Let X be a nonempty set and R a binary relation on X. The symmetric closure of R, denoted by R^s, is defined to be the set $R\cup R^{-1}$ by

$$R^s := R\cup R^{-1}. \tag{2.1}$$

Definition 2.3 [12] Let X be a nonempty set and R a binary relation on X. A sequence $\{x_n\}\subset X$ is called R-preserving if

$$(x_n,x_{n+1})\in R \quad \forall\, n\in\mathbb{N}_0. \tag{2.2}$$

Definition 2.4 [13] Let X be a nonempty set and T a self-mapping on X. A binary relation R on X is called T-closed if for all $x,y\in X$,

$$(x,y)\in R \Rightarrow (Tx,Ty)\in R. \tag{2.3}$$

Definition 2.5 [14] Let (X,d) be a metric space and R a binary relation on X. We say that (X,d) is R-complete if every R-preserving Cauchy sequence in X converges.

Definition 2.7 [15] Let (X,d) be a metric space. A binary relation R on X is called d-self-closed if for any R-preserving sequence $\{x_n\}$ such that $x_n\to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k},x]\in R$ for all $k\in\mathbb{N}_0$.

Definition 2.8 [16] Let X be a nonempty set and R a binary relation on X. A subset E of X is called R-directed if for each pair $x,y\in E$, there exists $z\in X$ such that $(x,z)\in R$ and $(y,z)\in R$.

We now introduce $\mathscr{F}$-metric space as follows:
Let $g:(0,+\infty)\to\mathbb{R}$ and $\mathscr{F}$ referred to the set of functions g satisfying:
$(\mathscr{F}_1)$ $0<x<t\Rightarrow g(x)\le g(t)$,
$(\mathscr{F}_2)$ for the sequence $\{x_n\}\subseteq\mathbb{R}^+$, $\lim_{n\to\infty}x_n=0\Leftrightarrow \lim_{n\to\infty}g(x_n)=-\infty$.

Definition 2.9 [17] Let X be a nonempty set and $d_{\mathscr{F}}:X\times X\to[0,+\infty)$. Suppose there exists $(g,h)\in\mathscr{F}\times[0,+\infty)$ such that
(i) $(x,y)\in X\times X$, $d_{\mathscr{F}}(x,y)=0\Leftrightarrow x=y$,
(ii) $d_{\mathscr{F}}(x,y)=d_{\mathscr{F}}(y,x)$, for all $(x,y)\in X\times X$,
(iii) For every $(x,y)\in X\times X$, for every $N\in\mathbb{N}$, $N\ge 2$, and for each $(u_i)_{i=1}^N\subset X$ with $(u_1,u_N)=(x,y)$, we have

$$D_{\mathscr{F}}(x,y)>0\Rightarrow g(d_{\mathscr{F}}(x,y))\le g\left[\sum_{i=1}^{N-1} d_{\mathscr{F}}(u_i,u_{i+1})\right]+h.$$

Then, $d_{\mathscr{F}}$ is referred to as an $\mathscr{F}$-metric on X and $(X,d_{\mathscr{F}})$ is called an $\mathscr{F}$-metric space.

Example 2.10 [18] Let $d_{\mathscr{F}}:\mathbb{R}\times\mathbb{R}\to[0,+\infty)$ be a function defined by

$$D_{\mathscr{F}}(x,y)=\begin{cases}(x-y)^2, & \text{if } (x,y)\in[0,3]\times[0,3]\\ |x-y|, & \text{if } (x,y)\notin[0,3]\times[0,3],\end{cases}$$

With $g(t)=\ln(t)$ and $h=\ln(3)$, is an $\mathscr{F}$-metric.

Definition 2.11 [19] Let $(X,d_{\mathscr{F}})$ be an $\mathscr{F}$-metric space.
(i) Let $\{x_n\}\subseteq X$. The sequence $\{x_n\}$ is referred to as $\mathscr{F}$-convergent to $x\in X$ if $\{x_n\}$ is convergent to x in $\mathscr{F}$-metric $d_{\mathscr{F}}$.
(ii) The sequence $\{x_n\}$ is referred to as $\mathscr{F}$-Cauchy, iff

$$\lim_{n,m\to\infty} d_{\mathscr{F}}(x_n,x_m)=0.$$

(iii) If each $\mathscr{F}$-Cauchy sequence in X is $\mathscr{F}$-convergent to $x\in X$, then $(X,d_{\mathscr{F}})$ is $\mathscr{F}$-complete.

3. Main Results

Theorem 3.1. Let $(X, d_{\mathscr{F}}, R)$ be a relational $\mathscr{F}$-metric space and T be a self mapping from X into X with the following conditions:
(i) $T(X) \subseteq Y \subseteq X$ such that $(Y, d_{\mathscr{F}})$ is R-complete,
(ii) $X[T,R]$ is nonempty,
(iii) R is T-closed;
(iv) Either T is R-continuous or $R|_Y$ is $d_{\mathscr{F}}$-self-closed;
(v) For all distinct $x, y \in X$ where $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + 2\beta + \gamma + \delta < 1$, such that

$$\begin{aligned} D_{\mathscr{F}}(Tx,Ty) \leq{}& \alpha d_{\mathscr{F}}(x,y) + \beta[d_{\mathscr{F}}(x,Tx) + d_{\mathscr{F}}(y,Ty)] \\ &+ \gamma \frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(x,Ty) + d_{\mathscr{F}}(y,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(y,Tx) + d_{\mathscr{F}}(x,Ty)} \\ &+ \delta \frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(x,y) + d_{\mathscr{F}}(x,Ty) + d_{\mathscr{F}}(y,Tx)} \end{aligned} \tag{3.1}$$

Then T has a fixed point in X.
Furthermore, if
(vi) $T(X)$ is R^s-connected, then T has a unique fixed point in X.
Additionally, the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathscr{F}$-convergent to a fixed point.

Proof. Using the hypothesis (ii), there exists $x_0 \in X[T,R]$ and construct sequence $\{x_n\} \subset X$ satisfying

$$X_{n+1} = Tx_n, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Since $(x_0, Tx_0) \in R$ and the T-closedness of R, we have

$$(Tx_0, T^2x_0), (T^2x_0, T^3x_0), \ldots, (T^n x_0, T^{n+1}x_0) \in R. \tag{3.3}$$

Using (3.2) and (3.3), we have

$$(x_n, x_{n+1}) \in R, \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Hence, $\{x_n\}$ is an R-preserving sequence. Using (3.1), (3.2) and (3.4), we have

$$\begin{aligned} D_{\mathscr{F}}(x_n,x_{n+1}) &= d_{\mathscr{F}}(Tx_{n-1},Tx_n) \\ &\leq \alpha d_{\mathscr{F}}(x_{n-1},x_n) + \beta[d_{\mathscr{F}}(x_{n-1},Tx_{n-1}) + d_{\mathscr{F}}(x_n,Tx_n)] \\ &\quad + \gamma \frac{d_{\mathscr{F}}(x_{n-1},Tx_{n-1})d_{\mathscr{F}}(x_{n-1},Tx_n) + d_{\mathscr{F}}(x_n,Tx_{n-1})d_{\mathscr{F}}(x_n,Tx_n)}{d_{\mathscr{F}}(x_n,Tx_{n-1}) + d_{\mathscr{F}}(x_{n-1},Tx_n)} \\ &\quad + \delta \frac{d_{\mathscr{F}}(x_{n-1},Tx_{n-1})d_{\mathscr{F}}(x_n,Tx_n)}{d_{\mathscr{F}}(x_{n-1},x_n) + d_{\mathscr{F}}(x_{n-1},Tx_n) + d_{\mathscr{F}}(x_n,Tx_{n-1})} \\ &= \alpha d_{\mathscr{F}}(x_{n-1},x_n) + \beta[d_{\mathscr{F}}(x_{n-1},x_n) + d_{\mathscr{F}}(x_n,x_{n+1})] \\ &\quad + \gamma d_{\mathscr{F}}(x_{n-1},x_n) + \delta d_{\mathscr{F}}(x_{n-1},x_n) \end{aligned}$$

This implies

$$(1-\beta)d_{\mathscr{F}}(x_n,x_{n+1}) \leq (\alpha+\beta+\gamma+\delta)d_{\mathscr{F}}(x_{n-1},x_n)$$

Which means

$$D_{\mathscr{F}}(x_n,x_{n+1}) \leq \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta}\right) d_{\mathscr{F}}(x_{n-1},x_n) \tag{3.5}$$

Let $\mu = \frac{\alpha+\beta+\gamma+\delta}{1-\beta}$ in (3.5), then we have

$$D_{\mathscr{F}}(x_n,x_{n+1}) \leq \mu d_{\mathscr{F}}(x_{n-1},x_n), \quad \forall n \in \mathbb{N}.$$

Similarly, we have by induction

$$D_{\mathscr{F}}(x_n,x_{n+1}) \leq \mu^n d_{\mathscr{F}}(x_0,x_1). \tag{3.6}$$

Using (3.6), $\forall m, n \in \mathbb{N}_0$ and $m \geq n$, we get

$$\begin{aligned} \sum_{i=n}^{m-1} d_{\mathscr{F}}(x_i,x_{i+1}) &\leq (\mu^n + \mu^{n+1} + \cdots + \mu^{m-1})d_{\mathscr{F}}(x_0,x_1) \\ &= \frac{\mu^n(1-\mu^{m-n})}{1-\mu} d_{\mathscr{F}}(x_0,x_1) \\ &\leq \frac{\mu^n}{1-\mu} d_{\mathscr{F}}(x_0,x_1) \to 0, \end{aligned}$$

As $n \to \infty$, that is, there exists $\varepsilon > 0$ fulfilling $0 < \frac{\mu^n}{1-\mu} d_{\mathscr{F}}(x_0,x_1) < \varepsilon$.

Let (g,h) fulfil (iii) of definition 2.9. By $(\mathscr{F}_2)$, for fixed $\varepsilon > 0$,

$$g\left(\frac{\mu^n}{1-\mu} d_{\mathscr{F}}(x_0,x_1)\right) < g(\varepsilon)–h.$$

Using (iii) of definition 2.9, $d_{\mathscr{F}}(x_n,x_m) > 0$ implies

$$G(d_{\mathscr{F}}(x_n,x_m)) \leq g\left(\sum_{i=n}^{m-1} d_{\mathscr{F}}(x_i,x_{i+1})\right) + h \leq g\left(\frac{\mu^n}{1-\mu} d_{\mathscr{F}}(x_0,x_1)\right) + h < g(\varepsilon).$$

Again, by $(\mathscr{F}_1)$

$$D_{\mathscr{F}}(x_n,x_m) < \varepsilon, \quad m \geq n.$$

Thus, $\{x_n\}$ is an $\mathscr{F}$-Cauchy sequence in X. Therefore, $\{x_n\}$ is an R-preserving $\mathscr{F}$-Cauchy sequence. Since $\{x_n\} \subseteq T(X) \subseteq Y$, hence $\{x_n\}$ is also an R-preserving $\mathscr{F}$-Cauchy sequence in Y. By hypothesis (i) $(Y, d_{\mathscr{F}})$ is R-complete, there exists x^* such that $x_n \to x^*$.
If T is R-continuous, then we have

$$X_{n+1} = Tx_n \to Tx^*.$$

So $Tx^* = x^*$, which shows x^* is a fixed point of T.
But if $R|_Y$ is $d_{\mathscr{F}}$-self-closed, then for the R-preserving sequence $\{x_n\}$ in Y as $x_n \to x^*$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x^*] \in R|_Y \subseteq R$, $\forall k \in \mathbb{N}_0$ and $x_{n_k+1} \to x^*$.
Now,

$$\begin{aligned} G(d_{\mathscr{F}}(x^*,Tx^*)) &\leq g(d_{\mathscr{F}}(x^*,Tx_{n_k}) + d_{\mathscr{F}}(Tx_{n_k},Tx^*)) + h \\ &= g(d_{\mathscr{F}}(x^*,x_{n_k+1}) + d_{\mathscr{F}}(x_{n_k+1},Tx^*)) + h \to -\infty \end{aligned}$$

As $n \to \infty$, using $(\mathscr{F}_2)$, this is a contradiction. Thus, $d_{\mathscr{F}}(x^*,Tx^*) = 0$, that is, $x^* = Tx^*$. This implies x^* is a fixed point of T.
To show the uniqueness of the fixed point, we let u and v be two distinct fixed points of T such that $Tu = u$ and $Tv = v$. So

$$U, v \in T(X) \text{ and } T^n u = u, \quad T^n v = v, \quad n \in \mathbb{N}_0.$$

There exists a path, say $\{z_1, z_2, z_3, \ldots, z_k\}$, of finite length k in R^s from u to v since $T(X)$ is R^s-connected, so that $z_0 = u$, $z_k = v$ and $(z_i, z_{i+1}) \in R^s$, $i = 0,1,2,\ldots,k-1$. By hypothesis (iii), R is T-closed, so $(T^n z_i, T^n z_{i+1}) \in R^s$, $i = 0,1,2,\ldots,k-1$ and $n \in \mathbb{N}_0$. Suppose $d_{\mathscr{F}}(u,v) > 0$, then

$$\begin{aligned} G(d_{\mathscr{F}}(u,v)) &= g(d_{\mathscr{F}}(T^n z_0, T^n z_k)) \\ &\leq g\left(\sum_{i=0}^{k-1} d_{\mathscr{F}}(T^n z_i, T^n z_{i+1})\right) + h \\ &\leq g\left(\mu \sum_{i=0}^{k-1} d_{\mathscr{F}}(T^{n-1} z_i, T^{n-1} z_{i+1})\right) + h \\ &\leq g\left(\mu^2 \sum_{i=0}^{k-1} d_{\mathscr{F}}(T^{n-2} z_i, T^{n-2} z_{i+1})\right) + h \\ &\ldots \\ &\leq g\left(\mu^n \sum_{i=0}^{k-1} d_{\mathscr{F}}(z_i, z_{i+1})\right) + h \to -\infty \end{aligned}$$

As $n \to \infty$, using $(\mathscr{F}_2)$, which is a contradiction. Thus, $d_{\mathscr{F}}(u,v) = 0$, that is, $u = v$. □

Corollary 3.2. Let $(X, d_{\mathscr{F}}, R)$ be a relational $\mathscr{F}$-metric space and T be a self mapping from X into X satisfy the following conditions:
(i) $T(X) \subseteq Y \subseteq X$ such that $(Y, d_{\mathscr{F}})$ is R-complete,
(ii) $X[T,R]$ is nonempty,
(iii) R is T-closed;
(iv) Either T is R-continuous or $R|_Y$ is $d_{\mathscr{F}}$-self-closed;
(v) For all distinct $x, y \in X$ with $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + 2\beta + \gamma + \delta < 1$, such that

$$\begin{aligned} D_{\mathscr{F}}(Tx,Ty) \leq\ & \alpha d_{\mathscr{F}}(x,y) + \beta[d_{\mathscr{F}}(x,Tx) + d_{\mathscr{F}}(y,Ty)] \\ & + \gamma \frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(x,Ty) + d_{\mathscr{F}}(y,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(y,Tx) + d_{\mathscr{F}}(x,Ty)} \\ & + \delta \frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(x,y) + d_{\mathscr{F}}(x,Ty) + d_{\mathscr{F}}(y,Tx)} \end{aligned} \tag{3.7}$$

Then T has a fixed point in X.
Furthermore, if
(vi) $R|_{T(X)}$ is complete, then T has a unique fixed point in X.
Additionally, the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathscr{F}$-convergent to a fixed point.

Proof. Adding to the proof of Theorem 3.1, if hypothesis (vi) holds, then for $x,y \in T(X)$ with $[x,y] \in R|_{T(X)}$ implies that $\{x,y\}$ is a path of length 1 in $R^s|_{T(X)}$ from x to y. Thus $T(X)$ is R^s-connected. Hence, we have the result. □

Corollary 3.3. Let $(X,d_{\mathscr{F}},R)$ be a relational $\mathscr{F}$-metric space and T be a self mapping from X into X satisfy the following conditions:
(i) $T(X) \subseteq Y \subseteq X$ such that $(Y,d_{\mathscr{F}})$ is R-complete,
(ii) $X[T,R]$ is nonempty,
(iii) R is T-closed;
(iv) Either T is R-continuous or $R|_Y$ is $d_{\mathscr{F}}$-self-closed;
(v) For all distinct $x,y \in X$ with $\alpha,\beta,\gamma,\delta \geq 0$ with $\alpha+2\beta+\gamma+\delta < 1$, such that

$$\begin{aligned} D_{\mathscr{F}}(Tx,Ty) \leq{}& \alpha d_{\mathscr{F}}(x,y) + \beta[d_{\mathscr{F}}(x,Tx)+d_{\mathscr{F}}(y,Ty)] \\ &+\gamma\frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(y,Tx)+d_{\mathscr{F}}(x,Ty)} \\ &+\delta\frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(x,y)+d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)} \end{aligned} \tag{3.8}$$

Then T has a fixed point in X.
Furthermore, if
(vi) $T(X)$ is R^s-directed, then T has a unique fixed point in X.
Additionally, the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathscr{F}$-convergent to a fixed point.

Proof. By adding to the proof of Theorem 3.1, if hypothesis (vi) holds, then for $x,y \in T(X)$, $\exists z \in X$ such that $(x,z) \in R^s$ and $(y,z) \in R^s$ implies that $\{x,y,z\}$ is a path of length 2 in R^s from x to y. Thus $T(X)$ is R^s-connected. Hence, we have the result. □

Corollary 3.4. Let $(X,d_{\mathscr{F}},R)$ be a relational $\mathscr{F}$-metric space and T be a self mapping from X into X satisfy the following conditions:
(i) $T(X) \subseteq Y \subseteq X$ such that $(Y,d_{\mathscr{F}})$ is R-complete,
(ii) $X[T,R]$ is nonempty,
(iii) R is T-closed;
(iv) Either T is R-continuous or $R|_Y$ is $d_{\mathscr{F}}$-self-closed;
(v) For all distinct $x,y \in X$ with $\alpha,\beta,\gamma,\delta \geq 0$ with $\alpha+2\beta+\gamma+\delta < 1$, such that

$$\begin{aligned} D_{\mathscr{F}}(Tx,Ty) \leq{}& \alpha d_{\mathscr{F}}(x,y) + \beta[d_{\mathscr{F}}(x,Tx)+d_{\mathscr{F}}(y,Ty)] \\ &+\gamma\frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(y,Tx)+d_{\mathscr{F}}(x,Ty)} \\ &+\delta\frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(x,y)+d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)} \end{aligned} \tag{3.9}$$

Then T has a fixed point in X.
Furthermore, if
(vi) $\mu(x,y,R^s) \neq \emptyset$, then T has a unique fixed point in X.
Additionally, the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathscr{F}$-convergent to a fixed point.

Proof. By adding to the proof of Theorem 3.1, if hypothesis (vi) holds, then for $x,y \in T(X)$, there exists a path, say $\{z_1,z_2,z_3,\ldots,z_k\}$, of finite length k in R^s from x to y such that $z_0 = x$, $z_k = y$. Thus $T(X)$ is R^s-connected. Hence, we have the result. □

Corollary 3.5. Let $(X,d_{\mathscr{F}},R)$ be a relational $\mathscr{F}$-metric space and T be a self mapping from X into X satisfy the following conditions:
(i) $T(X) \subseteq Y \subseteq X$ such that $(Y,d_{\mathscr{F}})$ is R-complete, (ii) $X[T,R]$ is nonempty,
(iii) R is T-closed;
(iv) Either T is R-continuous or $R|_Y$ is $d_{\mathscr{F}}$-self-closed;
(v) For all distinct $(x,y) \in R$ with $\mu \in (0,1)$, such that

$$D_{\mathscr{F}}(Tx,Ty) \leq \mu d_{\mathscr{F}}(x,y) \tag{3.10}$$

Then T has a fixed point in X.
Furthermore, if
(vi) $R|_{T(X)}$ is complete, then T has a unique fixed point in X.
Additionally, the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathscr{F}$-convergent to a fixed point.

Proof. The proof follows from Theorem 3.1. □

Example 3.6. Let $X = (-5,5)$ be the set with an $\mathscr{F}$-metric defined as

$$D_{\mathscr{F}}(x,y) = \begin{cases} 2|x–y|, & x \neq y \\ 0, & x = y \end{cases}$$

With $g(t) = -1/t$ and $h = 1$. Observe that $(X,d_{\mathscr{F}})$ is not a standard metric space. We now define a relation R on X as $R = \{(x,y) \in X \times X : x \geq y\}$ with symmetric closure as $R^s = \{(x,y) : \text{either } x \leq y \text{ or } y \leq x\}$.

Let $T : X \to X$ be defined by

$$T(x) = \begin{cases} x/5, & x \in (-5,0) \\ 0, & x \in [0,5). \end{cases}$$

If $Y = [-2,2) \subseteq X$, $T(X) = [-1,0]$. Then $T(X) \subseteq Y \subseteq X$ and $(Y, d_{\mathscr{F}})$ is R-complete. But X is neither complete nor R-complete. Clearly, R is T-closed, T is R-continuous or $R|_Y$ is $d_{\mathscr{F}}$-self-closed and $T(X)$ is R^s-connected. To check for the uniqueness of the fixed point, let $\mu \in [1/(2s), 1)$ and we have the uniqueness of the fixed point at 0. Additionally, if the sequence $\{x_n\} \subseteq X$ is such that $x_n = -1/n$, $n \in \mathbb{N}_0$, then it $\mathscr{F}$-converges to 0.

Theorem 3.7. Let $(X, d_{\mathscr{F}}, R)$ be a relational $\mathscr{F}$-metric space and T be a self mapping from X into X satisfy the following conditions:
(i) $T(X) \subseteq Y \subseteq X$ such that $(Y, d_{\mathscr{F}})$ is R-complete,
(ii) $X[T,R]$ is nonempty,
(iii) R is T-closed;
(iv) Either T is R-continuous or $R|_Y$ is $d_{\mathscr{F}}$-self-closed;
(v) For all distinct $x, y \in X$ with $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + 2\beta + \gamma + \delta < 1$, such that

$$\begin{aligned} D_{\mathscr{F}}(Tx,Ty) \leq{}& \alpha d_{\mathscr{F}}(x,y) + \beta[d_{\mathscr{F}}(x,Ty) + d_{\mathscr{F}}(y,Tx)] \\ &+ \gamma \frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(x,Ty) + d_{\mathscr{F}}(y,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(y,Tx) + d_{\mathscr{F}}(x,Ty)} \\ &+ \delta \frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(x,y) + d_{\mathscr{F}}(x,Ty) + d_{\mathscr{F}}(y,Tx)} \end{aligned} \tag{3.11}$$

Then T has a fixed point in X.
Furthermore, if
(vi) $T(X)$ is R^s-connected, then T has a unique fixed point in X.
Additionally, the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathscr{F}$-convergent to a fixed point.

Proof. Using the hypothesis (ii), there exists $x_0 \in X[T,R]$ and construct sequence $\{x_n\} \subset X$ satisfying

$$X_{n+1} = Tx_n, \quad \forall n \in \mathbb{N}. \tag{3.12}$$

Since $(x_0, Tx_0) \in R$ and the T-closedness of R, we have

$$(Tx_0, T^2x_0), (T^2x_0, T^3x_0), \ldots, (T^nx_0, T^{n+1}x_0) \in R. \tag{3.13}$$

Using (3.12) and (3.13), we have

$$(x_n, x_{n+1}) \in R, \quad \forall n \in \mathbb{N}. \tag{3.14}$$

Hence, $\{x_n\}$ is an R-preserving sequence. Using (3.11), (3.12) and (3.14), we have

$$\begin{aligned} D_{\mathscr{F}}(x_n,x_{n+1}) &= d_{\mathscr{F}}(Tx_{n-1},Tx_n) \\ &\leq \alpha d_{\mathscr{F}}(x_{n-1},x_n) + \beta[d_{\mathscr{F}}(x_{n-1},Tx_n) + d_{\mathscr{F}}(x_n,Tx_{n-1})] \\ &\quad + \gamma \frac{d_{\mathscr{F}}(x_{n-1},Tx_{n-1})d_{\mathscr{F}}(x_{n-1},Tx_n) + d_{\mathscr{F}}(x_n,Tx_{n-1})d_{\mathscr{F}}(x_n,Tx_n)}{d_{\mathscr{F}}(x_n,Tx_{n-1}) + d_{\mathscr{F}}(x_{n-1},Tx_n)} \\ &\quad + \delta \frac{d_{\mathscr{F}}(x_{n-1},Tx_{n-1})d_{\mathscr{F}}(x_n,Tx_n)}{d_{\mathscr{F}}(x_{n-1},x_n) + d_{\mathscr{F}}(x_{n-1},Tx_n) + d_{\mathscr{F}}(x_n,Tx_{n-1})} \\ &= \alpha d_{\mathscr{F}}(x_{n-1},x_n) + \beta[d_{\mathscr{F}}(x_{n-1},x_{n+1}) + d_{\mathscr{F}}(x_n,x_n)] \\ &\quad + \gamma d_{\mathscr{F}}(x_{n-1},x_n) + \delta d_{\mathscr{F}}(x_{n-1},x_n) \end{aligned}$$

By the triangular inequality, we have

$$(1-\beta)d_{\mathscr{F}}(x_n,x_{n+1}) \leq (\alpha+\beta+\gamma+\delta)d_{\mathscr{F}}(x_{n-1},x_n)$$

Which implies

$$D_{\mathscr{F}}(x_n,x_{n+1}) \leq \left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta}\right) d_{\mathscr{F}}(x_{n-1},x_n) \tag{3.15}$$

Let $\mu = \frac{\alpha+\beta+\gamma+\delta}{1-\beta}$ in (3.15), then we have

$$D_{\mathscr{F}}(x_n,x_{n+1}) \leq \mu d_{\mathscr{F}}(x_{n-1},x_n), \quad \forall n \in \mathbb{N}.$$

Similarly, we have by induction

$$D_{\mathscr{F}}(x_n,x_{n+1}) \leq \mu^n d_{\mathscr{F}}(x_0,x_1). \tag{3.16}$$

Using (3.16), $\forall m,n \in \mathbb{N}_0$ and $m \geq n$, we get

$$\begin{aligned}\sum_{i=n}^{m-1} d_{\mathscr{F}}(x_i,x_{i+1}) &\leq (\mu^n+\mu^{n+1}+\cdots+\mu^{m-1})d_{\mathscr{F}}(x_0,x_1)\\ &= \frac{\mu^n(1-\mu^{m-n})}{1-\mu}d_{\mathscr{F}}(x_0,x_1)\\ &\leq \frac{\mu^n}{1-\mu}d_{\mathscr{F}}(x_0,x_1) \to 0,\end{aligned}$$

As $n \to \infty$, that is, there exists $\varepsilon > 0$ fulfilling $0 < \frac{\mu^n}{1-\mu}d_{\mathscr{F}}(x_0,x_1) < \varepsilon$.
Let (g,h) fulfil (iii) of definition 2.9. By $(\mathscr{F}_2)$, for fixed $\varepsilon > 0$,

$$g\left(\frac{\mu^n}{1-\mu}d_{\mathscr{F}}(x_0,x_1)\right) < g(\varepsilon)–h. \tag{3.17}$$

Using (iii) of definition 2.9, $d_{\mathscr{F}}(x_n,x_m) > 0$ implies

$$G(d_{\mathscr{F}}(x_n,x_m)) \leq g\left(\sum_{i=n}^{m-1} d_{\mathscr{F}}(x_i,x_{i+1})\right)+h \leq g\left(\frac{\mu^n}{1-\mu}d_{\mathscr{F}}(x_0,x_1)\right)+h < g(\varepsilon).$$

Again, by $(\mathscr{F}_1)$

$$D_{\mathscr{F}}(x_n,x_m) < \varepsilon, \quad m \geq n.$$

Thus, $\{x_n\}$ is an $\mathscr{F}$-Cauchy sequence in X. The rest of the proof follows exactly the same arguments as the proof of Theorem 3.1, establishing that x^* is the unique fixed point in X. □

Corollary 3.8. Let $(X,d_{\mathscr{F}},R)$ be a relational $\mathscr{F}$-metric space and T be a self mapping from X into X satisfy the following conditions:
(i) $T(X) \subseteq Y \subseteq X$ such that $(Y,d_{\mathscr{F}})$ is R-complete,
(ii) $X[T,R]$ is nonempty,
(iii) R is T-closed;
(iv) Either T is R-continuous or $R|_Y$ is $d_{\mathscr{F}}$-self-closed;
(v) For all distinct $x,y \in X$ with $\alpha,\beta,\gamma,\delta \geq 0$ with $\alpha+2\beta+\gamma+\delta < 1$, such that

$$\begin{aligned}D_{\mathscr{F}}(Tx,Ty) \leq{}& \alpha d_{\mathscr{F}}(x,y)+\beta[d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)]\\ &+\gamma\frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(y,Tx)+d_{\mathscr{F}}(x,Ty)}\\ &+\delta\frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(x,y)+d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)}\end{aligned} \tag{3.18}$$

Then T has a fixed point in X.
Furthermore, if
(vi) $R|_{T(X)}$ is complete, then T has a unique fixed point in X.
Additionally, the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathscr{F}$-convergent to a fixed point.

Proof. Adding to the proof of Theorem 3.7, if hypothesis (vi) holds, then for $x,y \in T(X)$, $[x,y] \in R|_{T(X)}$ implies that $\{x,y\}$ is a path of length 1 in $R^s|_{T(X)}$ from x to y. Thus $T(X)$ is R^s-connected. Hence, we have the result. □

Corollary 3.9. Let $(X,d_{\mathscr{F}},R)$ be a relational $\mathscr{F}$-metric space and T be a self mapping from X into X satisfy the following conditions:
(i) $T(X) \subseteq Y \subseteq X$ such that $(Y,d_{\mathscr{F}})$ is R-complete,
(ii) $X[T,R]$ is nonempty,
(iii) R is T-closed;
(iv) Either T is R-continuous or $R|_Y$ is $d_{\mathscr{F}}$-self-closed;
(v) For all distinct $x,y \in X$ with $\alpha,\beta,\gamma,\delta \geq 0$ with $\alpha+2\beta+\gamma+\delta < 1$, such that

$$\begin{aligned}D_{\mathscr{F}}(Tx,Ty) \leq{}& \alpha d_{\mathscr{F}}(x,y)+\beta[d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)]\\ &+\gamma\frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(y,Tx)+d_{\mathscr{F}}(x,Ty)}\\ &+\delta\frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(x,y)+d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)}\end{aligned} \tag{3.19}$$

Then T has a fixed point in X.
Furthermore, if
(vi) $T(X)$ is R^s-directed, then T has a unique fixed point in X.
Additionally, the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathscr{F}$-convergent to a fixed point.

Proof. By adding to the proof of Theorem 3.7, if hypothesis (vi) holds, then $x,y \in T(X)$, $\exists z \in X$ such that $(x,z) \in R^s$ and $(y,z) \in R^s$ implies that $\{x,y,z\}$ is a path of length 2 in R^s from x to y. Thus $T(X)$ is R^s-connected. Hence, we have the result. □

Corollary 3.10. Let $(X,d_{\mathscr{F}},R)$ be a relational $\mathscr{F}$-metric space and T be a self mapping from X into X satisfy the following conditions:
(i) $T(X) \subseteq Y \subseteq X$ such that $(Y,d_{\mathscr{F}})$ is R-complete,
(ii) $X[T,R]$ is nonempty,
(iii) R is T-closed;
(iv) Either T is R-continuous or $R|_Y$ is $d_{\mathscr{F}}$-self-closed;
(v) For all distinct $x,y \in X$ with $\alpha,\beta,\gamma,\delta \geq 0$ with $\alpha+2\beta+\gamma+\delta<1$, such that

$$\begin{aligned} D_{\mathscr{F}}(Tx,Ty) \leq{} & \alpha d_{\mathscr{F}}(x,y)+\beta[d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)] \\ & +\gamma\frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(y,Tx)+d_{\mathscr{F}}(x,Ty)} \\ & +\delta\frac{d_{\mathscr{F}}(x,Tx)d_{\mathscr{F}}(y,Ty)}{d_{\mathscr{F}}(x,y)+d_{\mathscr{F}}(x,Ty)+d_{\mathscr{F}}(y,Tx)} \end{aligned} \tag{3.20}$$

Then T has a fixed point in X.
Furthermore, if
(vi) $\mu(x,y,R^s) \neq \emptyset$, then T has a unique fixed point in X.
Additionally, the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathscr{F}$-convergent to a fixed point.

Proof. By adding to the proof of Theorem 3.7, if hypothesis (vi) holds, then for $x,y \in T(X)$, there exists a path, say $\{z_1,z_2,z_3,\ldots,z_k\}$, of finite length k in R^s from x to y such that $z_0 = x$, $z_k = y$. Thus $T(X)$ is R^s-connected. Hence, we have the result. ☐

4. Application

In this section, we apply our previous results to solve boundary value problems. Let $I=[0,1]$ and $C[I,\mathbb{R}]$ be the collection of all continuous functions on $[0,1]$ and $X=C[I,\mathbb{R}]$. We now consider the following Theorem.
Theorem 4.1. Consider the second order differential equation

$$\frac{d^2x}{dt^2} = \phi(t,x(t)), \quad \text{with} \quad x(0)=0,\ x(1)=0. \tag{4.1}$$

Where $t \in I$ and $\phi : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function.
If there exists $\lambda \geq 0$ such that

$$0 \leq \phi(t,x(t))+\lambda x(t)-[\phi(t,y(t))+\lambda y(t)] \leq \lambda\,\frac{x(t)-y(t)}{1+x(t)-y(t)}, \tag{4.2}$$

Where $x(t),y(t) \in X$ and $x(t) \geq y(t)$, then there exists a unique solution x^* in X of the boundary value problem (4.1).

Proof. The boundary value problem in (4.1) can be rewritten as

$$X'(t)+\lambda x(t) = \phi(t,x(t))+\lambda x(t), \quad t\in[0,1],\ x(0)=0,\ x(1)=0.$$

Or

$$X(t) = \int_0^1 G(t,\xi)[\phi(\xi,x(\xi))+\lambda x(\xi)]\,d\xi, \quad \text{for } t \in I, \tag{4.3}$$

Where the Green function is given by

$$G(t,\xi) = \begin{cases} (1-t)\xi, & 0 \leq \xi \leq t \leq 1 \\ (1-\xi)t, & 0 \leq t \leq \xi \leq 1 \end{cases}$$

Define $T: X \to X$ such that $x \in X$ is a solution of (4.3) if and only if it is the solution of (4.1). Then

$$Tx(t) = \int_0^1 G(t,\xi)[\phi(\xi,x(\xi))+\lambda x(\xi)]\,d\xi$$

With a binary relation

$$R = \{(x,y) \in X\times X : x(t),y(t) \geq 0 \text{ with } (x-y)(t) \geq 0, \forall t \in I\}.$$

Then, we have the following cases:
Case I. Now consider

$$D_{\mathscr{F}}(x,y) = \begin{cases} \exp\left(\sup_{t\in I}|x(t)-y(t)|\right), & x \neq y \\ 0, & x=y, \end{cases}$$

Where $g(t) = -1/t$, $t>0$ and $h=1$. Then, $(X,d_{\mathscr{F}})$ is an R-complete $\mathscr{F}$-metric space.
Case II. Choose an R-preserving Cauchy sequence $\{x_n(t)\}$ such that $x_n(t) \to x(t)$. Then, we have $x_n(t)x_{n+1}(t) \geq 0$ and $x_n(t) \geq x_{n+1}(t)$, for $t \in I$, $n \in \mathbb{N}_0$. Then, we have either $x_n(t) \geq 0$ or $x_n(t) \leq 0$. Assuming $x_n(t) \geq 0$ for $t \in I$, we have a positive sequence converging to $x(t)$. Thus, $x(t) \geq 0$, that is, $(x_n(t),x(t)) \in R$, for $t \in I$, $n \in \mathbb{N}_0$. This implies R is $d_{\mathscr{F}}$-self-closed.

Case III. For $(x,y) \in R$, that is, $x(t) \geq y(t)$ by (4.2), $\phi(t,y(t)) + \lambda y(t) \leq \phi(t,x(t)) + \lambda x(t)$, $\forall t \in I$ and $G(t,\xi) \geq 0$, $(t,\xi) \in I \times I$, we obtain

$$\begin{aligned} Tx(t) &= \int_0^1 G(t,\xi)[\phi(\xi,x(\xi)) + \lambda x(\xi)]\,d\xi \\ &\geq \int_0^1 G(t,\xi)[\phi(\xi,y(\xi)) + \lambda y(\xi)]\,d\xi = Ty(t), \end{aligned}$$

This implies $(Tx(t),Ty(t)) \in R$, that is, R is T-closed. For any $x(t) \geq 0$, $Tx(t) \geq 0$, that is, $(x(t),Tx(t)) \in R$. Then, $X[T,R]$ is non-empty.
Case IV. For $(x,y) \in R$,

$$\begin{aligned} D_{\mathscr{F}}(Tx,Ty) &= \exp\left(\sup_{t\in I}|Tx(t)–Ty(t)|\right) \\ &= \exp\left(\sup_{t\in I}\left|\int_0^1 G(t,\xi)[\phi(\xi,x(\xi)) + \lambda x(\xi)]\,d\xi - \int_0^1 G(t,\xi)[\phi(\xi,y(\xi)) + \lambda y(\xi)]\,d\xi\right|\right) \\ &\leq \exp\left(\sup_{t\in I}\int_0^1 G(t,\xi)\lambda\,\frac{x(\xi)–y(\xi)}{1+x(\xi)–y(\xi)}\,d\xi\right) \\ &\leq \exp\left(\sup_{\xi\in I}|x(\xi)–y(\xi)|\int_0^1 G(t,\xi)\,d\xi\right) \\ &= \exp\left(\sup_{\xi\in I}|x(\xi)–y(\xi)|\left(\int_0^t (1–t)\xi\,d\xi + \int_t^1 (1-\xi)t\,d\xi\right)\right) \\ &\leq \exp\left(\sup_{t\in I}|x(t)–y(t)|\left(\frac{1}{8}\right)\right) \\ &\leq \mu\exp\left(\sup_{t\in I}|x(t)–y(t)|\right) \\ &= \mu d_{\mathscr{F}}(x,y), \quad \text{for } \mu \in [0,1). \end{aligned}$$

If $X = Y = C[I,\mathbb{R}]$, then Y is R^s-connected. Hence, all the hypotheses of Theorem 3.1 are satisfied and T has a unique fixed point. $\square$

5. Conclusion

In this article, we utilized the notion of $\mathscr{F}$-metric spaces and obtained some fixed point results for relation-theoretic contractive type mapping in complete $\mathscr{F}$-metric spaces. This study provides significant advancements in the understanding of relation-theoretic mapping, through illustrative numerical examples, we showcased the practical applicability of the results and explored as an application, the solution for two points boundary value problems. Future work could also explore the extension of these results to other types of relation-theoretic mapping.

Article Information

Acknowledgment: The authors want to thank everyone who has assisted us in finishing this task.

Conflicting interests: The authors declared that there are no competing interests.

Disclaimer (Artificial Intelligence): The author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.), and text-to-image generators have been used during writing or editing of manuscripts.

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