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Research Article

Nonlinear Contraction-Based Coupled Fixed Point Theorems in Partial b-Metric Spaces with Applications

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Abstract

In (Mediterr. J. Math. 11, 703–711, 2014), Shukla introduced the concept of partial b-metric spaces as a generalization of partial metric and b-metric spaces. The purpose of this article is to establish some coupled fixed point results for contractive mappings in a partial b-metric space. Examples are provided to support the results and concepts presented herein. As an application of our results, we prove the well-posedness of coupled fixed point problems.

1. Introduction

Banach's fixed point theorems for contraction mappings is one of the pivotal results of mathematical analysis. Banach contraction principle [1] is one of the most notable results which has played a vital role in the development of a metric fixed point theory. This principle and its variants provide a useful apparatus in guaranteeing the existence and uniqueness of solutions of various nonlinear problems: differential equations, integral equations, optimization problems, variational inequalities. Since its introduction, the Banach contraction mapping principle has been generalized and refined in numerous ways, leading to a wealth of articles dedicated to its improvement [2–7].

Matthews [8] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification. In the mentioned papers, Mathews studied topological properties of partial metric spaces and stated a modified version of a Banach contraction mapping principle on the space. Existence of fixed points in partial ordered metric spaces was first investigated by Turinici [9]. Ran and Reurings [10] studied the existence of fixed points for certain mappings in partially ordered metric spaces and applied their results to matrix equations. Their results were later extended by Nieto and Lopez [11] for non decreassing mappings and obtained the solutions of certain partial differential equations with periodic boundary conditions. The concept of mixed monotone property for contractive operators of the form $F : X \to X \times X$ was first introduced by Bhaskar and Lakshmikantham [12] where X is a partially ordered metric, and then established some coupled fixed point theorems.

Czerwik [13] had an excellent idea to introduced the concepts of b-metric spaces by weakening the coefficient of the triangle inequality and generalized Banach's Contraction Principle. Subsequently, Boriceanu [14] obtained some concrete examples of b-metric spaces and studied the fixed-point properties of set-valued operators in b-metric spaces.

Aydi [15] introduced some coupled fixed point results for mappings satisfying different contractive conditions on complete partial metric spaces. Later on, Shukla [16] introduced partial b-metric spaces as a generalization of both b-metric spaces and partial metric spaces. Moreover, he proved Banach contraction principle as well as the Kannan type fixed point theorem in partial b-metric spaces. Recently, Saluja [17] established some coupled fixed point results for contractive type conditions in the framework of partial b-metric space.

In view of the above considerations, we prove some coupled fixed point results for contractive mappings in a partial b-metric space. Examples are provided to support results and concepts presented herein. As an application of our results, we prove well-posedness of coupled fixed point problems.

Throughout the article, we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers and by $\mathbb{N}$ the set of natural numbers.

2. Preliminaries

We start this section by presenting some relevant definitions and lemma.

Definition 2.1 [18] Let X be a non-empty set and $s \geq 1$ be a given real number. If a function $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:

(b1) $d(x,y) = 0$ iff $x = y$
(b2) $d(x,y) = d(y,x)$
(b3) $d(x,y) \leq s[d(x,z) + d(z,y)]$ for all $x,y \in X$,

then the pair (X,d) is called a b-metric space.

Remark 2.2 The class of b-metric spaces is larger than the class of metric spaces since any metric space is a b-metric space with constant $s = 1$. Therefore, it is obvious that b-metric spaces generalize metric spaces.

We can present an example, which show that introducing a b-metric space instead of a metric space is meaningful since there exist b-metric spaces which are not metric spaces.

Example 2.3 Let $X = [0,\infty)$ and $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = |x-y|^p,$$

where p is a real number such that $p > 1$. Let $x,y,z \in X$. By taking $u = x-z$ and $v = z-y$, we have

$$\begin{aligned} |x-y|^p &= |u+v|^p \leq (|u|+|v|)^p \\ &\leq (2\max\{|u|,|v|\})^p \\ &\leq 2^p(|x-z|^p + |z-y|^p) \end{aligned}$$

which implies that

$$d(x,y) \leq 2^p[d(x,z) + d(z,y)].$$

Therefore (X,d) is a b-metric space with coefficient 2^p. On the other hand, for $x > z > y$, we have

$$\begin{aligned} |x-y|^p &= |u+v|^p = (u+v)^p > u^p + v^p \\ &= (x-z)^p + (z-y)^p = |x-z|^p + |z-y|^p, \end{aligned}$$

which implies that

$$d(x,y) > d(x,z) + d(z,y).$$

Therefore (X,d) is not a metric space.

Definition 2.4 [19, 20]Let X be a nonempty set. A mapping

$$p : X \times X \to \mathbb{R}^+$$

is said to be a partial metric on X if for all $x,y,z \in X$ the following conditions are satisfied:

(p1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;
(p2) $p(x,x) \leq p(x,y)$;
(p3) $p(x,y) = p(y,x)$;
(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

In this case, the pair (X,p) is called a partial metric space.
The most basic example of a partial metric space is the pair $(\mathbb{R}^+,p)$, where

$$p(x,y)=\max\{x,y\}$$

for all $x,y\in\mathbb{R}^+$.

Definition 2.5 [16] A partial b-metric on a nonempty set X is a function

$$p_b : X\times X\to\mathbb{R}^+$$

such that for all $x,y,z\in X$:

(pb1) $x=y$ if and only if $p_b(x,x)=p_b(x,y)=p_b(y,y)$;
(pb2) $p_b(x,x)\le p_b(x,y)$;
(pb3) $p_b(x,y)=p_b(y,x)$;
(pb4) there exists a real number $s\ge 1$ such that

$$p_b(x,y)\le s[p_b(x,z)+p_b(z,y)-p_b(z,z)].$$

A partial b-metric space is a pair (X,p_b) such that X is a nonempty set and p_b is a partial b-metric on X. The number s is called the coefficient of (X,p_b).

Remark 2.6 In a partial b-metric space (X,p_b) if $x,y\in X$ and $p_b(x,y)=0$, then $x=y$, but the converse may not be true.

Remark 2.7 It is clear that every partial metric space is a partial b-metric space with coefficient $s=1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.
The following example shows that the converse of these facts need not hold.

Example 2.8 Let $X=\{0,1,2\}$ and define

$$p_b : X\times X\to\mathbb{R}^+$$

as follows:

$$p_b(0,0)=0,\quad p_b(1,1)=4,\quad p_b(0,1)=p_b(1,0)=8,$$

$$p_b(2,2)=2,\quad p_b(0,2)=p_b(2,0)=5,\quad p_b(1,2)=p_b(2,1)=4.$$

Because $p_b(2,2)=2\neq 0$. Thus, condition (b1) is not established, therefore (X,p_b) is not a b-metric space. Since

$$8=p_b(0,1)>p_b(0,2)+p_b(2,1)-p_b(2,2)=5+4-2=7,$$

thus, condition (pb4) is not established, therefore (X,p_b) is not a partial metric space. But, we know (X,p_b) is a partial b-metric space with constant $s\ge 2$.

Definition 2.9 [16] Let (X,p_b) be a partial b-metric space with coefficient s. Then:

(i) A sequence $\{x_n\}$ in (X,p_b) is said to be convergent with respect to τ_{p_b} and converges to a point $x\in X$, if

$$\lim_{n\to\infty} p_b(x_n,x)=p_b(x,x).$$

(ii) A sequence $\{x_n\}$ is said to be a Cauchy sequence in (X,p_b) if

$$\lim_{n,m\to\infty} p_b(x_n,x_m)$$

exists and is finite.

(iii) (X,p_b) is said to be a complete partial b-metric space if for every Cauchy sequence $\{x_n\}$ in X there exists $x\in X$ such that

$$\lim_{n,m\to\infty} p_b(x_n,x_m)=\lim_{n\to\infty} p_b(x_n,x)=p_b(x,x).$$

(iv) A mapping $g:X\to X$ is said to be continuous at $x_0\in X$ if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$g\big(B_{p_b}(x_0,\delta)\big)\subset B_{p_b}\big(g(x_0),\varepsilon\big).$$

Note. In a partial b-metric space the limit of a convergent sequence may not be unique.

Example 2.10 [16] Let $X=[0,+\infty)$, $c>0$ be any constant and define

$$p_b : X\times X\to[0,+\infty)$$

by

$$p_b(x,y)=\max\{x,y\}+c$$

for all $x, y \in X$.
Then (X, p_b) is a partial b-metric space with arbitrary coefficient $s \geq 1$. Now define a sequence $\{x_n\}$ in X by

$$x_n = 1 \quad \text{for all } n \in \mathbb{N}.$$

Note that, if $y \geq 1$, we have

$$p_b(x_n, y) = y + c = p_b(y, y),$$

therefore

$$\lim_{n\to\infty} p_b(x_n, y) = p_b(y, y)$$

for all $y \geq 1$. Thus, the limit of a convergent sequence in a partial b-metric space need not be unique.

Definition 2.11 [21] An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping

$$f : X \times X \to X$$

if

$$f(x, y) = x \quad \text{and} \quad f(y, x) = y.$$

Example 2.12 [22] Let $X = [0, +\infty)$ and $f : X \times X \to X$ be defined by

$$f(x, y) = \frac{x + y}{6}$$

for all $x, y \in X$. One can easily see that f has a unique coupled fixed point $(0, 0)$.

Example 2.13 [22] Let $X = [0, +\infty)$ and $f : X \times X \to X$ be defined by

$$f(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then we see that f has two coupled fixed points $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

3. Main Results

In this section, we begin with the following theorem.

Theorem 3.1 Let (X, p_b) be a complete partial b-metric space with constant $s \geq 1$. Let

$$f : X \times X \to X$$

be a mapping satisfying the following condition:

$$p_b\big(f(x,y), f(u,v)\big) \leq \frac{\lambda}{s^2} \max\left\{p_b(x,u),\, p_b(f(x,y),x),\, p_b(f(u,v),u),\, \frac{p_b(f(u,v),u)\,[1 + p_b(f(x,y),x)]}{1 + p_b(x,u)}\right\} \tag{3.1}$$

for all $x, y, u, v \in X$ and $\lambda \in (0, 1)$. Then f has a unique coupled fixed point in X and $p_b(a, a) = 0$ for some $a \in X$.

Proof. Let $x_0, y_0 \in X$ be arbitrary points in X. Let

$$x_1 = f(x_0, y_0), \qquad y_1 = f(y_0, x_0).$$

Again let

$$x_2 = f(x_1, y_1), \qquad y_2 = f(y_1, x_1).$$

Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ of points in X such that

$$x_{n+1} = f(x_n, y_n), \qquad y_{n+1} = f(y_n, x_n), \qquad \text{for all } n \in \mathbb{N}.$$

Now, by using (3.1) and $(pb2)$, $(pb3)$ of Definition 2.5, we get

$$p_b(x_n, x_{n+1}) = p_b\big(f(x_{n-1}, y_{n-1}), f(x_n, y_n)\big) \tag{1}$$

$$\leq \frac{\lambda}{s^2} \max\left\{p_b(x_{n-1}, x_n),\, p_b\big(f(x_{n-1}, y_{n-1}), x_{n-1}\big),\, p_b\big(f(x_n, y_n), x_n\big),\, \frac{p_b(f(x_n, y_n), x_n)\,[1 + p_b(f(x_{n-1}, y_{n-1}), x_{n-1})]}{1 + p_b(x_{n-1}, x_n)}\right\} \tag{2}$$

$$= \frac{\lambda}{s^2} \max\left\{p_b(x_{n-1}, x_n),\, p_b(x_n, x_{n-1}),\, p_b(x_{n+1}, x_n),\, \frac{p_b(x_{n+1}, x_n)\,[1 + p_b(x_n, x_{n-1})]}{1 + p_b(x_{n-1}, x_n)}\right\} \tag{3}$$

$$= \frac{\lambda}{s^2} \max\left\{p_b(x_{n-1}, x_n),\, p_b(x_{n+1}, x_n)\right\}. \tag{3.2}$$

Similarly, we get

$$\begin{aligned} p_b(y_n,y_{n+1}) &= p_b\big(f(y_{n-1},x_{n-1}),f(y_n,x_n)\big) \qquad (4)\\ &\le \frac{\lambda}{s^2}\max\left\{p_b(y_{n-1},y_n),\, p_b(y_{n+1},y_n)\right\}. \qquad (3.3)\end{aligned}$$

Set

$$A_n = p_b(x_n,x_{n+1}) \quad \text{and} \quad B_n = p_b(y_n,y_{n+1}) \qquad (3.4)$$

$$\mu_1 = \max\{A_{n-1},A_n\} \qquad (3.5)$$

and

$$\mu_2 = \max\{B_{n-1},B_n\}. \qquad (3.6)$$

Again, set

$$\begin{aligned} C_n &= A_n + B_n \qquad (5)\\ &= p_b(x_n,x_{n+1}) + p_b(y_n,y_{n+1}). \qquad (3.7)\end{aligned}$$

From (3.2)–(3.7), we have

$$C_n \le \frac{\lambda}{s^2}(\mu_1+\mu_2). \qquad (3.8)$$

If $\mu_1 = A_n$ and $\mu_2 = B_n$, then from (3.8) we have

$$C_n \le \frac{\lambda}{s^2}(A_n+B_n) = \frac{\lambda}{s^2}C_n,$$

that is,

$$\left(1-\frac{\lambda}{s^2}\right)C_n \le 0,$$

a contradiction, since $\lambda \in (0,1)$ and $s \ge 1$.
Hence,

$$C_n \le \frac{\lambda}{s^2}(A_{n-1}+B_{n-1}) = \frac{\lambda}{s^2}C_{n-1}. \qquad (3.9)$$

Then, by repeated application of (3.9) for each $n \in \mathbb{N}$, we have

$$C_n \le \frac{\lambda}{s^2}C_{n-1} \le \left(\frac{\lambda}{s^2}\right)^2 C_{n-2} \le \cdots \le \left(\frac{\lambda}{s^2}\right)^n C_0. \qquad (3.10)$$

If $C_0 = 0$, then

$$p_b(x_0,x_1) + p_b(y_0,y_1) = 0.$$

Hence, we get

$$x_0 = x_1 = f(x_0,y_0) \quad \text{and} \quad y_0 = y_1 = f(y_0,x_0),$$

which implies that (x_0,y_0) is a coupled fixed point of f.
Now, assume $C_0 > 0$. For each $n > m$ where $n,m \in \mathbb{N}$, we have by using condition $(pb4)$ of Definition 2.5

$$\begin{aligned} p_b(x_n,x_m) &\le s\big[p_b(x_n,x_{n+1}) + p_b(x_{n+1},x_m)\big] - p_b(x_{n+1},x_{n+1})\\ &\le s\,p_b(x_n,x_{n+1}) + s^2\big[p_b(x_{n+1},x_{n+2}) + p_b(x_{n+2},x_m)\big] - s\,p_b(x_{n+2},x_{n+2})\\ &\le s\,p_b(x_n,x_{n+1}) + s^2\,p_b(x_{n+1},x_{n+2}) + s^3\,p_b(x_{n+2},x_{n+3}) + \cdots + s^{m-n}\,p_b(x_{m-1},x_m). \qquad (3.11)\end{aligned}$$

Similarly, we have

$$p_b(y_n,y_m) \le s\,p_b(y_n,y_{n+1}) + s^2\,p_b(y_{n+1},y_{n+2}) + s^3\,p_b(y_{n+2},y_{n+3}) + \cdots + s^{m-n}\,p_b(y_{m-1},y_m). \qquad (3.12)$$

Adding (3.11) and (3.12), we get

$$\begin{aligned} p_b(x_n,x_m) + p_b(y_n,y_m) &\le s\,C_n + s^2\,C_{n+1} + s^3\,C_{n+2} + \cdots + s^{m-n}\,C_{m-1}\\ &\le \left[s\left(\frac{\lambda}{s^2}\right)^n + s^2\left(\frac{\lambda}{s^2}\right)^{n+1} + s^3\left(\frac{\lambda}{s^2}\right)^{n+2} + \cdots + s^{m-n}\left(\frac{\lambda}{s^2}\right)^{m-1}\right]C_0\\ &= \left[\sum_{i=1}^{m-n} s^i\left(\frac{\lambda}{s^2}\right)^{i+n-1}\right]C_0 = \left[\sum_{t=1}^{m-1} s^t\left(\frac{\lambda}{s^2}\right)^t\right]C_0\\ &\le \left[\sum_{t=1}^{n}\left(\frac{\lambda}{s^2}\right)^t\right]C_0 = \frac{\left(\frac{\lambda}{s^2}\right)^n}{1-\frac{\lambda}{s^2}}\,C_0. \qquad (3.13)\end{aligned}$$

As $\lambda \in (0,1/s)$ and $s \geq 1$, from (3.13) we get

$$p_b(x_n,x_m)+p_b(y_n,y_m) \to 0 \quad \text{as } n \to \infty.$$

This proves that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X,p_b).
Since (X,p_b) is complete, there exist $x^*,y^* \in X$ such that

$$x_n \to x^*, \quad y_n \to y^* \quad \text{as } n \to \infty,$$

and

$$p_b(x^*,x^*) = \lim_{n\to\infty} p_b(x_n,x^*) = \lim_{n,m\to\infty} p_b(x_n,x_m) = 0, \tag{3.14}$$

$$p_b(y^*,y^*) = \lim_{n\to\infty} p_b(y_n,y^*) = \lim_{n,m\to\infty} p_b(y_n,y_m) = 0. \tag{3.15}$$

Now, we prove that

$$x^* = f(x^*,y^*) \quad \text{and} \quad y^* = f(y^*,x^*).$$

Arguing by contradiction, assume that $x^* \neq f(x^*,y^*)$ and $y^* \neq f(y^*,x^*)$, so that

$$D_1 = p_b(x^*,f(x^*,y^*)) > 0 \quad \text{and} \quad D_2 = p_b(y^*,f(y^*,x^*)) > 0.$$

Then

$$\begin{aligned}
D_1 &= p_b(x^*,f(x^*,y^*)) \\
&\leq p_b(x^*,x_{n+1}) + p_b(x_{n+1},f(x^*,y^*)) - p_b(x_{n+1},x_{n+1}) \\
&\leq p_b(x^*,x_{n+1}) + p_b(x_{n+1},f(x^*,y^*)) \\
&= p_b(x_{n+1},f(x^*,y^*)) + p_b(x^*,x_{n+1}) \\
&= p_b(f(x_n,y_n),f(x^*,y^*)) + p_b(x^*,x_{n+1}) \\
&\leq \frac{\lambda}{s^2} \max\left\{ p_b(x_n,x^*),\, p_b(f(x_n,y_n),x_n),\, p_b(f(x^*,y^*),x^*),\, \frac{p_b(f(x^*,y^*),x^*)\,[1+p_b(f(x_n,y_n),x_n)]}{1+p_b(x_n,x^*)} \right\} \\
&= \frac{\lambda}{s^2} \max\left\{ p_b(x_n,x^*),\, p_b(x_{n+1},x_n),\, p_b(f(x^*,y^*),x^*),\, \frac{p_b(f(x^*,y^*),x^*)\,[1+p_b(x_{n+1},x_n)]}{1+p_b(x_n,x^*)} \right\}.
\end{aligned} \tag{3.16}$$

On taking the limit $n \to \infty$ in (3.16) and using (3.14), we get

$$D_1 \leq \frac{\lambda}{s^2} p_b(x^*,f(x^*,y^*)) = \frac{\lambda}{s^2} D_1,$$

a contradiction, since $\lambda \in (0,1)$ and $s \geq 1$. Hence, $D_1 = 0$, that is,

$$p_b(x^*,f(x^*,y^*)) = 0 \quad \text{and therefore} \quad x^* = f(x^*,y^*).$$

Similarly, we can show that

$$y^* = f(y^*,x^*).$$

Thus, (x^*,y^*) is a coupled fixed point of f.
To show uniqueness of the coupled fixed point, suppose that (w,z) is another coupled fixed point of f such that $(x^*,y^*) \neq (w,z)$. Then from (3.1), (3.14), (3.15), and condition $(pb3)$ of Definition 2.5, we have

$$\begin{aligned}
p_b(x^*,w) &= p_b(f(x^*,y^*),f(w,z)) \\
&\leq \frac{\lambda}{s^2} \max\left\{ p_b(x^*,w),\, p_b(f(x^*,y^*),x^*),\, p_b(f(w,z),w),\, \frac{p_b(f(w,z),w)\,[1+p_b(f(x^*,y^*),x^*)]}{1+p_b(x^*,w)} \right\} \\
&= \frac{\lambda}{s^2} \max\left\{ p_b(x^*,w),\, p_b(x^*,y^*),\, p_b(w,w),\, \frac{p_b(w,w)\,[1+p_b(x^*,x^*)]}{1+p_b(x^*,w)} \right\} \\
&= \frac{\lambda}{s^2} \max\{p_b(x^*,w),0,0,0\} \\
&= \frac{\lambda}{s^2} p_b(x^*,w).
\end{aligned} \tag{3.17}$$

a contradiction, since $\lambda \in (0,1)$ and $s \geq 1$. Hence, we have

$$p_b(x^*,w) = 0 \quad \text{and so} \quad x^* = w.$$

Similarly, we can show that $y^* = z$. Thus, f has a unique coupled fixed point in X.

Theorem 3.2 Let (X, p_b) be a complete partial b-metric space with constant $s \geq 1$. Let

$$f : X \times X \to X$$

be a mapping satisfying the following condition:

$$p_b(f(x,y), f(u,v)) \leq a_1 p_b(x,u) + a_2 p_b(y,v) + a_3 p_b(f(x,y),x) + a_4 p_b(f(u,v),u) + a_5 p_b(f(x,y),u) + a_6 p_b(f(u,v),x) + a_7 \frac{p_b(f(u,v),u)\,[1 + p_b(f(x,y),x)]}{1 + p_b(x,u)} \tag{3.18}$$

for all $x, y, u, v \in X$, where $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \geq 0$ such that

$$a_1 + a_2 + a_3 + a_4 + a_5 + 2sa_6 + a_7 < 1.$$

Then f has a unique coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be any arbitrary points in X. Define

$$x_1 = f(x_0,y_0), \quad y_1 = f(y_0,x_0), \quad x_2 = f(x_1,y_1), \quad y_2 = f(y_1,x_1),$$

and continue this process to construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = f(x_n,y_n), \quad y_{n+1} = f(y_n,x_n), \quad \forall n \in \mathbb{N}.$$

Set

$$\alpha_n = p_b(x_n,x_{n+1}), \quad \beta_n = p_b(y_n,y_{n+1}).$$

Now, using (3.18) and $(pb2)$, $(pb3)$ of Definition 2.5, we get

$$\begin{aligned}
p_b(x_n,x_{n+1}) &= p_b(f(x_{n-1},y_{n-1}), f(x_n,y_n)) \\
&\leq a_1 p_b(x_{n-1},x_n) + a_2 p_b(y_{n-1},y_n) + a_3 p_b(f(x_{n-1},y_{n-1}),x_{n-1}) + a_4 p_b(f(x_n,y_n),x_n) \\
&\quad + a_5 p_b(f(x_{n-1},y_{n-1}),x_n) + a_6 p_b(f(x_n,y_n),x_{n-1}) + a_7 \frac{p_b(f(x_n,y_n),x_n)[1 + p_b(f(x_{n-1},y_{n-1}),x_{n-1})]}{1 + p_b(x_{n-1},x_n)} \\
&= a_1 p_b(x_{n-1},x_n) + a_2 p_b(y_{n-1},y_n) + a_3 p_b(x_n,x_{n-1}) + a_4 p_b(x_{n+1},x_n) \\
&\quad + a_5 p_b(x_n,x_n) + a_6 p_b(x_{n+1},x_{n-1}) + a_7 \frac{p_b(x_{n+1},x_n)[1 + p_b(x_n,x_{n-1})]}{1 + p_b(x_{n-1},x_n)} \\
&\leq a_1 p_b(x_{n-1},x_n) + a_2 p_b(y_{n-1},y_n) + a_3 p_b(x_n,x_{n-1}) + a_4 p_b(x_{n+1},x_n) \\
&\quad + a_5 p_b(x_n,x_{n+1}) + a_6 \left[s(p_b(x_{n+1},x_n) + p_b(x_n,x_{n-1})) - p_b(x_n,x_n)\right] \\
&\quad + a_7 \frac{p_b(x_{n+1},x_n)[1 + p_b(x_n,x_{n-1})]}{1 + p_b(x_{n-1},x_n)} \\
&\leq a_1 p_b(x_{n-1},x_n) + a_2 p_b(y_{n-1},y_n) + a_3 p_b(x_n,x_{n-1}) + a_4 p_b(x_{n+1},x_n) \\
&\quad + a_5 p_b(x_n,x_{n+1}) + a_6 s[p_b(x_{n+1},x_n) + p_b(x_n,x_{n-1})] + a_7 p_b(x_{n+1},x_n) \\
&= (a_1 + a_3 + sa_6)\, p_b(x_{n-1},x_n) + (a_4 + a_5 + sa_6 + a_7)\, p_b(x_{n+1},x_n) + a_2\, p_b(y_{n-1},y_n).
\end{aligned} \tag{3.19}$$

Since $\alpha_n = p_b(x_n,x_{n+1})$ and $\beta_n = p_b(y_n,y_{n+1})$, from (3.19) we have

$$\alpha_n \leq (a_1 + a_3 + sa_6)\, \alpha_{n-1} + (a_4 + a_5 + sa_6 + a_7)\, \alpha_n + a_2\, \beta_{n-1}. \tag{3.20}$$

Similarly, we obtain

$$\begin{aligned}
p_b(y_n,y_{n+1}) &= p_b(f(y_{n-1},x_{n-1}), f(y_n,x_n)) \\
&\leq (a_1 + a_3 + sa_6)\, p_b(y_{n-1},y_n) + (a_4 + a_5 + sa_6 + a_7)\, p_b(y_{n+1},y_n) + a_2\, p_b(x_{n-1},x_n).
\end{aligned} \tag{3.21}$$

It follows that

$$\beta_n \leq (a_1 + a_3 + sa_6)\, \beta_{n-1} + (a_4 + a_5 + sa_6 + a_7)\, \beta_n + a_2\, \alpha_{n-1}. \tag{3.22}$$

Set

$$\gamma_n = \alpha_n + \beta_n. \tag{3.23}$$

Adding (3.20) and (3.22), we get

$$\begin{aligned}
\gamma_n &= \alpha_n + \beta_n \\
&\leq (a_1 + a_3 + sa_6)\, (\alpha_{n-1} + \beta_{n-1}) + (a_4 + a_5 + sa_6 + a_7)\, (\alpha_n + \beta_n) + a_2 (\alpha_{n-1} + \beta_{n-1}) \\
&= (a_1 + a_3 + sa_6)\, \gamma_{n-1} + (a_4 + a_5 + sa_6 + a_7)\, \gamma_n + a_2\, \gamma_{n-1} \\
&= (a_1 + a_2 + a_3 + sa_6)\, \gamma_{n-1} + (a_4 + a_5 + sa_6 + a_7)\, \gamma_n.
\end{aligned} \tag{3.24}$$

It follows that

$$\gamma_n \le \frac{a_1+a_2+a_3+sa_6}{1-(a_4+a_5+sa_6+a_7)}\,\gamma_{n-1}. \tag{3.25}$$

Let

$$t = \frac{a_1+a_2+a_3+sa_6}{1-(a_4+a_5+sa_6+a_7)}.$$

Since $a_1+a_2+a_3+a_4+a_5+2sa_6+a_7<1$, it follows that $t<1$. Hence,

$$\gamma_n \le t\,\gamma_{n-1}, \quad \forall n\in\mathbb{N}. \tag{3.26}$$

Repeating the application of (3.26) for all $n\in\mathbb{N}$, we have

$$\gamma_n \le t\,\gamma_{n-1} \le t^2\,\gamma_{n-2} \le \cdots \le t^n\,\gamma_0. \tag{3.27}$$

If $\gamma_0=0$, that is, $p_b(x_0,x_1)+p_b(y_0,y_1)=0$, then we have

$$x_0=x_1=f(x_0,y_0) \quad \text{and} \quad y_0=y_1=f(y_0,x_0),$$

which implies that (x_0,y_0) is a coupled fixed point of f.
Now, assume $\gamma_0>0$. For each $n>m$ with $n,m\in\mathbb{N}$, using condition $(pb4)$ of Definition 2.5, we get

$$\begin{aligned}
p_b(x_n,x_m) &\le s\big[p_b(x_n,x_{n+1})+p_b(x_{n+1},x_m)\big]-p_b(x_{n+1},x_{n+1})\\
&\le s\,p_b(x_n,x_{n+1})+s^2\big[p_b(x_{n+1},x_{n+2})+p_b(x_{n+2},x_m)\big]-s\,p_b(x_{n+2},x_{n+2})\\
&\le s\,p_b(x_n,x_{n+1})+s^2\,p_b(x_{n+1},x_{n+2})+s^3\,p_b(x_{n+2},x_{n+3})+\cdots+s^{m-n}\,p_b(x_{m-1},x_m).
\end{aligned} \tag{3.28}$$

Similarly,

$$p_b(y_n,y_m) \le s\,p_b(y_n,y_{n+1})+s^2\,p_b(y_{n+1},y_{n+2})+s^3\,p_b(y_{n+2},y_{n+3})+\cdots+s^{m-n}\,p_b(y_{m-1},y_m). \tag{3.29}$$

Adding (3.28) and (3.29), we get

$$\begin{aligned}
p_b(x_n,x_m)+p_b(y_n,y_m) &\le s\,\gamma_n+s^2\,\gamma_{n+1}+s^3\,\gamma_{n+2}+\cdots+s^{m-n}\,\gamma_{m-1}\\
&\le \big[s\,t^n+s^2\,t^{n+1}+s^3\,t^{n+2}+\cdots+s^{m-n}\,t^{m-1}\big]\gamma_0\\
&= \Big[\sum_{r=1}^{m-n} s^r t^{r+n-1}\Big]\gamma_0 = \Big[\sum_{p=1}^{m-1} s^p t^p\Big]\gamma_0\\
&\le \Big[\sum_{p=1}^{n} (st)^p\Big]\gamma_0 = \frac{(st)^n}{1-st}\,\gamma_0.
\end{aligned} \tag{3.30}$$

As $t\in(0,1)$ and $s\ge 1$, it follows from (3.30) that

$$p_b(x_n,x_m)+p_b(y_n,y_m)\to 0 \quad \text{as } n\to\infty.$$

This proves that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X,p_b). Since (X,p_b) is complete, there exist $x^*,y^*\in X$ such that $x_n\to x^*$, $y_n\to y^*$ as $n\to\infty$ and

$$p_b(x^*,x^*) = \lim_{n\to\infty} p_b(x_n,x^*) = \lim_{n,m\to\infty} p_b(x_n,x_m) = 0, \tag{3.31}$$

$$p_b(y^*,y^*) = \lim_{n\to\infty} p_b(y_n,y^*) = \lim_{n,m\to\infty} p_b(y_n,y_m) = 0. \tag{3.32}$$

Now, we show that $x^*=f(x^*,y^*)$ and $y^*=f(y^*,x^*)$. Arguing by contradiction, assume that $x^*\neq f(x^*,y^*)$ and $y^*\neq f(y^*,x^*)$, so that

$$M=p_b(x^*,f(x^*,y^*))>0 \quad \text{and} \quad N=p_b(y^*,f(y^*,x^*))>0.$$

Then,

$$\begin{aligned}
M &= p_b(x^*,f(x^*,y^*))\\
&\le p_b(x^*,x_{n+1})+p_b(x_{n+1},f(x^*,y^*))-p_b(x_{n+1},x_{n+1})\\
&\le p_b(x^*,x_{n+1})+p_b(x_{n+1},f(x^*,y^*))\\
&= p_b(x_{n+1},f(x^*,y^*))+p_b(x^*,x_{n+1})\\
&= p_b(f(x_n,y_n),f(x^*,y^*))+p_b(x^*,x_{n+1})\\
&\le a_1p_b(x_n,x^*)+a_2p_b(y_n,y^*)+a_3p_b(f(x_n,y_n),x_n)+a_4p_b(f(x^*,y^*),x^*)\\
&\quad +a_5p_b(f(x_n,y_n),x^*)+a_6p_b(f(x^*,y^*),x_n)+a_7\frac{p_b(f(x^*,y^*),x^*)[1+p_b(f(x_n,y_n),x_n)]}{1+p_b(x_n,x^*)}+p_b(x^*,x_{n+1})\\
&= a_1p_b(x_n,x^*)+a_2p_b(y_n,y^*)+a_3p_b(x_{n+1},x_n)+a_4p_b(f(x^*,y^*),x^*)+a_5p_b(x_{n+1},x^*)\\
&\quad +a_6p_b(f(x^*,y^*),x_n)+a_7\frac{p_b(f(x^*,y^*),x^*)[1+p_b(x_{n+1},x_n)]}{1+p_b(x_n,x^*)}+p_b(x^*,x_{n+1}).
\end{aligned} \tag{3.33}$$

On taking the limit as $n \to \infty$ in (3.33) and using (3.31), we obtain

$$M \leq (a_4 + a_6 + a_7)\, p_b(x^*, f(x^*, y^*)) = (a_4 + a_6 + a_7)M. \tag{3.34}$$

This is a contradiction, since $a_4 + a_6 + a_7 < 1$. Hence, $M = 0$, that is,

$$p_b(x^*, f(x^*, y^*)) = 0,$$

and therefore

$$x^* = f(x^*, y^*).$$

Similarly, we can show that $y^* = f(y^*, x^*)$. Thus, (x^*, y^*) is a coupled fixed point of f.

To show the uniqueness of the coupled fixed point, suppose that (w, z) is another coupled fixed point of f such that $(x^*, y^*) \neq (w, z)$. Then from (3.18), (3.31), (3.32) and condition $(pb3)$ of Definition 2.5, we have

$$\begin{aligned}
p_b(x^*, w) &= p_b(f(x^*, y^*), f(w, z)) \\
&\leq a_1 p_b(x^*, w) + a_2 p_b(y^*, z) + a_3 p_b(f(x^*, y^*), x^*) + a_4 p_b(f(w, z), w) \\
&\quad + a_5 p_b(f(x^*, y^*), w) + a_6 p_b(f(w, z), x^*) + a_7 \frac{p_b(f(w, z), w)\,[1 + p_b(f(x^*, y^*), x^*)]}{1 + p_b(x^*, w)} \\
&= a_1 p_b(x^*, w) + a_2 p_b(y^*, z) + a_3 p_b(x^*, x^*) + a_4 p_b(w, w) \\
&\quad + a_5 p_b(x^*, w) + a_6 p_b(w, x^*) + a_7 \frac{p_b(w, w)\,[1 + p_b(x^*, x^*)]}{1 + p_b(x^*, w)} \\
&= (a_1 + a_5 + a_6)\, p_b(x^*, w) + a_2\, p_b(y^*, z).
\end{aligned} \tag{3.35}$$

Similarly,

$$\begin{aligned}
p_b(y^*, z) &= p_b(f(y^*, x^*), f(z, w)) \\
&\leq (a_1 + a_5 + a_6) p_b(y^*, z) + a_2 p_b(x^*, w).
\end{aligned} \tag{3.36}$$

Adding (3.35) and (3.36), we get

$$\begin{aligned}
p_b(x^*, w) + p_b(y^*, z) &\leq (a_1 + a_5 + a_6)[p_b(x^*, w) + p_b(y^*, z)] + a_2[p_b(x^*, w) + p_b(y^*, z)] \\
&= (a_1 + a_2 + a_5 + a_6)[p_b(x^*, w) + p_b(y^*, z)].
\end{aligned} \tag{3.37}$$

This is a contradiction, since $a_1 + a_2 + a_5 + a_6 < 1$. Hence, we have

$$p_b(x^*, w) + p_b(y^*, z) = 0 \quad \text{and so} \quad x^* = w \text{ and } y^* = z.$$

Thus, f has a unique coupled fixed point in X.

Corollary 3.3 Let (X, p_b) be a complete partial b-metric space with constant $s \geq 1$. Let

$$f : X \times X \to X$$

be a mapping satisfying the following condition:

$$\begin{aligned}
p_b(f(x, y), f(u, v)) \leq{}& a_1\, p_b(x, u) + a_2\, p_b(y, v) + a_3\, p_b(f(x, y), x) + a_4\, p_b(f(u, v), u) \\
&+ a_5\, p_b(f(x, y), u) + a_6\, p_b(f(u, v), x)
\end{aligned} \tag{3.38}$$

for all $x, y, u, v \in X$, where $a_1, a_2, a_3, a_4, a_5, a_6 \geq 0$ such that

$$a_1 + a_2 + a_3 + a_4 + a_5 + 2sa_6 < 1.$$

Then f has a unique coupled fixed point in X.

Corollary 3.4 Let (X, p_b) be a complete partial b-metric space with constant $s \geq 1$. Let

$$f : X \times X \to X$$

be a mapping satisfying the following condition:

$$p_b(f(x, y), f(u, v)) \leq a_1\, p_b(x, u) + a_2\, p_b(y, v) \tag{3.39}$$

for all $x, y, u, v \in X$, where $a_1, a_2 \geq 0$ such that

$$a_1 + a_2 < 1.$$

Then f has a unique coupled fixed point in X.

Proof Let $a_3 = a_4 = a_5 = a_6 = a_7 = 0$ in Theorem 3.2. Then the result follows immediately.

Corollary 3.5 Let (X, p_b) be a complete partial b-metric space with constant $s \geq 1$. Let

$$f : X \times X \to X$$

be a mapping satisfying the following condition:

$$p_b(f(x,y), f(u,v)) \leq \frac{\lambda}{s^2} \max\left\{p_b(x,u),\, p_b(f(x,y),x),\, p_b(f(u,v),u)\right\} \tag{3.40}$$

for all $x, y, u, v \in X$ and $\lambda \in (0,1)$. Then f has a unique coupled fixed point in X and $p_b(a,a) = 0$ for some $a \in X$.

Example 3.6 Let $X = [0, +\infty)$. Let p be a real number such that $p > 1$, and let X be endowed with the usual partial b-metric p_b defined by

$$p_b : X \times X \to [0, +\infty), \qquad p_b(x,y) = [\max\{x,y\}]^p + |x-y|,$$

for all $x, y \in X$. Then (X, p_b) is a partial b-metric space with arbitrary coefficient $s = 2^p > 1$.
Consider the mapping

$$f : X \times X \to X, \qquad f(x,y) = \frac{x+y}{6}.$$

Now, for any $x, y, u, v \in X$,

$$\begin{aligned}
p_b(f(x,y), f(u,v)) &= \left[\max\{f(x,y), f(u,v)\}\right]^p + \left|f(x,y) - f(u,v)\right| \\
&= \left[\max\left\{\frac{x+y}{6}, \frac{u+v}{6}\right\}\right]^p + \left|\frac{x+y}{6} - \frac{u+v}{6}\right| \\
&= \left[\frac{1}{6}\max\{x+y, u+v\}\right]^p + \frac{1}{6}|(x+y) - (u+v)| \\
&\leq \left[\frac{1}{6}\max\{x,u\} + \frac{1}{6}\max\{y,v\}\right]^p + \frac{1}{6}|(x-u) + (y-v)| \\
&\leq \left[\frac{1}{6}\max\{x,u\}\right]^p + \left[\frac{1}{6}\max\{y,v\}\right]^p + \frac{1}{6}|x-u| + \frac{1}{6}|y-v| \\
&\leq \frac{1}{6}\Big([\max\{x,u\}]^p + |x-u|\Big) + \frac{1}{6}\Big([\max\{y,v\}]^p + |y-v|\Big) \\
&= a_1\, p_b(x,u) + a_2\, p_b(y,v),
\end{aligned}$$

Let $X = [0, +\infty)$ and $p > 1$. Define a partial b-metric $p_b : X \times X \to [0, +\infty)$ by

$$p_b(x,y) = [\max\{x,y\}]^p + |x-y|, \quad \forall x, y \in X.$$

Then (X, p_b) is a partial b-metric space with coefficient $s = 2^p > 1$.
Consider the mapping $f : X \times X \to X$ defined by $f(x,y) = \frac{x+y}{6}$. For any $x, y, u, v \in X$, we have

$$\begin{aligned}
p_b(f(x,y), f(u,v)) &= \left[\max\{f(x,y), f(u,v)\}\right]^p + |f(x,y) - f(u,v)| \\
&= \left[\max\left\{\frac{x+y}{6}, \frac{u+v}{6}\right\}\right]^p + \frac{1}{6}|(x+y) - (u+v)| \\
&\leq \left[\frac{1}{6}\max\{x,u\} + \frac{1}{6}\max\{y,v\}\right]^p + \frac{1}{6}|x-u| + \frac{1}{6}|y-v| \\
&\leq \frac{1}{6}\big([\max\{x,u\}]^p + |x-u|\big) + \frac{1}{6}\big([\max\{y,v\}]^p + |y-v|\big) \\
&= a_1\, p_b(x,u) + a_2\, p_b(y,v),
\end{aligned}$$

where $a_1 = a_2 = \frac{1}{6}$ and $a_1 + a_2 = \frac{1}{3} < 1$.
Hence, by Corollary 3.4, f has a unique coupled fixed point $(0,0)$ in X.
Now, consider $f : X \times X \to X$ given by $f(x,y) = \frac{x+y}{2}$. Then for $x, y, u, v \in X$,

$$\begin{aligned}
p_b(f(x,y), f(u,v)) &= \left[\max\{f(x,y), f(u,v)\}\right]^p + |f(x,y) - f(u,v)| \\
&= \left[\max\left\{\frac{x+y}{2}, \frac{u+v}{2}\right\}\right]^p + \frac{1}{2}|(x+y) - (u+v)| \\
&\leq \frac{1}{2}\big([\max\{x,u\}]^p + |x-u|\big) + \frac{1}{2}\big([\max\{y,v\}]^p + |y-v|\big) \\
&= a_1\, p_b(x,u) + a_2\, p_b(y,v),
\end{aligned}$$

where $a_1 = a_2 = \frac{1}{2}$ and $a_1 + a_2 = 1$.
In this case, both $(0,0)$ and $(1,1)$ are coupled fixed points of f, so the coupled fixed point is *not unique*.

4. Application

In this section, as an application of our results, we establish that Theorems 3.1 and 3.2 can be utilized to derive the *well-posedness* of the coupled fixed point problem in partial b-metric spaces. We begin with the following definitions and theorem.

Definition 4.1 [23] Let (X,d) be a metric space and $f:X\to X$ be a mapping. The problem of finding a fixed point of f is said to be *well-posed* if:

1. f has a unique fixed point x_0,
2. for any sequence $\{x_n\}\subset X$ with $\lim_{n\to\infty} d(fx_n,x_n)=0$, we have

$$\lim_{n\to\infty} d(x_n,x_0)=0.$$

Now, we extend the concept to coupled fixed points in partial b-metric spaces.
Let $C_0\mathrm{FP}(f,X\times X)$ denote a *coupled fixed point problem* of a mapping f, and let $C_0F(f)$ denote the set of all coupled fixed points of f:

$$C_0F(f)=\{(x,y)\in X\times X \mid f(x,y)=x,\ f(y,x)=y\}.$$

A coupled fixed point problem $C_0\mathrm{FP}(f,X\times X)$ is said to be *well-posed* if:

1. f has a unique coupled fixed point $(x^*,y^*)\in X\times X$,
2. for any sequences $\{x_n\},\{y_n\}\subset X$ with

$$\lim_{n\to\infty} p_b(f(x_n,y_n),x_n)=0 \quad \text{and} \quad \lim_{n\to\infty} p_b(f(y_n,x_n),y_n)=0,$$

we have

$$\lim_{n\to\infty} p_b(x_n,x^*)=0 \quad \text{and} \quad \lim_{n\to\infty} p_b(y_n,y^*)=0.$$

Definition 4.2 [17] Let (X,p_b) be a partial b-metric space with coefficient $s\geq 1$, and let $f:X\times X\to X$ be a mapping. The coupled fixed point problem $C_0\mathrm{FP}(f,X\times X)$ is said to be *well-posed* if:

1. $C_0F(f)$ is unique,
2. for any sequences $\{x_n\},\{y_n\}\subset X$ with $(x^*,y^*)\in C_0F(f)$ and

$$\lim_{n\to\infty} p_b(f(x_n,y_n),x_n)=0=\lim_{n\to\infty} p_b(f(y_n,x_n),y_n),$$

we have

$$x^*=\lim_{n\to\infty} x_n, \quad y^*=\lim_{n\to\infty} y_n. \tag{4.1}$$

Theorem 4.3

Let (X,p_b) be a complete partial b-metric space with coefficient $s\geq 1$, and let $f:X\times X\to X$ be a mapping satisfying the conditions of Theorem 3.2. For any sequences $\{x_n\},\{y_n\}\subset X$ with $(x^*,y^*)\in C_0F(f)$, if

$$\lim_{n\to\infty} p_b(f(x_n,y_n),x_n)=0=\lim_{n\to\infty} p_b(f(y_n,x_n),y_n), \tag{4.2}$$

then the coupled fixed point problem of f is well-posed, and $p_b(x^*,x^*)=0$ for some $x^*\in X$.
Proof

By Theorem 3.2, the mapping f has a unique coupled fixed point $(x_0,y_0)\in X\times X$. Let $\{x_n\},\{y_n\}\subset X$ be sequences such that

$$\lim_{n\to\infty} p_b(f(x_n,y_n),x_n)=0=\lim_{n\to\infty} p_b(f(y_n,x_n),y_n).$$

Assume that $(x_0,y_0)\neq(x_n,y_n)$ for any nonnegative integer n. Using $f(x_0,y_0)=x_0$ and $f(y_0,x_0)=y_0$, we have

$$\begin{aligned}
p_b(x_0,x_n) &\leq p_b(f(x_0,y_0),f(x_n,y_n))+p_b(f(x_n,y_n),x_n)-p_b(f(x_n,y_n),f(x_n,y_n))\\
&\leq p_b(f(x_0,y_0),f(x_n,y_n))+p_b(f(x_n,y_n),x_n)\\
&\leq a_1p_b(x_0,x_n)+a_2p_b(y_0,y_n)+a_3p_b(f(x_0,y_0),x_0)+a_4p_b(f(x_n,y_n),x_n)\\
&\quad +a_5p_b(f(x_0,y_0),x_n)+a_6p_b(f(x_n,y_n),x_0)\\
&\quad +a_7\frac{p_b(f(x_n,y_n),x_n)\,[1+p_b(f(x_0,y_0),x_0)]}{1+p_b(x_0,x_n)}+p_b(f(x_n,y_n),x_n).
\end{aligned}$$

Now, using the fact that (x_0,y_0) is a coupled fixed point, $p_b(f(x_0,y_0),x_0)=0$, and applying the properties of the partial b-metric, inequality (**??**) reduces to

$$p_b(x_0,x_n)\leq(a_1+a_5+a_6)p_b(x_0,x_n)+a_2p_b(y_0,y_n)+(a_4+a_7+1)p_b(f(x_n,y_n),x_n).$$

Similarly, we obtain

$$p_b(y_0,y_n)\leq(a_1+a_5+a_6)p_b(y_0,y_n)+a_2p_b(x_0,x_n)+(a_4+a_7+1)p_b(f(y_n,x_n),y_n).$$

Adding (4.5) and (4.6), we get

$$p_b(x_0,x_n)+p_b(y_0,y_n) \leq (a_1+a_2+a_5+a_6)\,[p_b(x_0,x_n)+p_b(y_0,y_n)]+\varepsilon_n,$$

where

$$\varepsilon_n = (a_4+a_7+1)\,[p_b(f(x_n,y_n),x_n)+p_b(f(y_n,x_n),y_n)] \to 0 \quad \text{as } n\to\infty.$$

Since $a_1+a_2+a_5+a_6<1$, taking the limit $n\to\infty$ yields

$$\lim_{n\to\infty}[p_b(x_0,x_n)+p_b(y_0,y_n)]=0,$$

which implies

$$\lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad \lim_{n\to\infty} y_n = y_0.$$

Thus, the coupled fixed point problem of f is well-posed and $p_b(x_0,x_0)=0$ for some $x_0\in X$.

$$\begin{aligned}
p_b(y_0,y_n) &\leq p_b(f(y_0,x_0),f(y_n,x_n))+p_b(f(y_n,x_n),y_n)-p_b(f(y_n,x_n),f(y_n,x_n))\\
&\leq p_b(f(y_0,x_0),f(y_n,x_n))+p_b(f(y_n,x_n),y_n)\\
&\leq a_1p_b(y_0,y_n)+a_2p_b(x_0,x_n)+a_3p_b(f(y_0,x_0),y_0)+a_4p_b(f(y_n,x_n),y_n)\\
&\quad +a_5p_b(f(y_0,x_0),y_n)+a_6p_b(f(y_n,x_n),y_0)\\
&\quad +a_7\frac{p_b(f(y_n,x_n),y_n)\,[1+p_b(f(y_0,x_0),y_0)]}{1+p_b(y_0,y_n)}+p_b(f(y_n,x_n),y_n)\\
&=(a_1+a_5+a_6)p_b(y_0,y_n)+a_2p_b(x_0,x_n)+(a_4+a_7+1)p_b(f(y_n,x_n),y_n).
\end{aligned}$$

Since

$$\lim_{n\to\infty} p_b(f(x_n,y_n),x_0)=0=\lim_{n\to\infty} p_b(f(y_n,x_n),y_0),$$

for $(x_0,y_0)\in C_0F(f)$, and using condition (pb3) of Definition 2.5 in (4.3) and (4.4), we get

$$\lim_{n\to\infty} p_b(x_0,x_n) \leq (a_1+a_5)\lim_{n\to\infty} p_b(x_0,x_n)+a_2\lim_{n\to\infty} p_b(y_0,y_n), \tag{4.5}$$

and

$$\lim_{n\to\infty} p_b(y_0,y_n) \leq (a_1+a_5)\lim_{n\to\infty} p_b(y_0,y_n)+a_2\lim_{n\to\infty} p_b(x_0,x_n). \tag{4.6}$$

Now, set

$$M=\lim_{n\to\infty} p_b(x_0,x_n) \quad \text{and} \quad N=\lim_{n\to\infty} p_b(y_0,y_n), \tag{4.7}$$

Adding (4.5) and (4.6), we have

$$\begin{aligned}
M+N &\leq (a_1+a_5)(M+N)+a_2(M+N)\\
&=(a_1+a_2+a_5)(M+N)\\
&<M+N,
\end{aligned} \tag{4.8}$$

since $a_1+a_2+a_5<1$. A contradiction arises, since $a_1+a_2+a_5<1$. Hence,

$$\lim_{n\to\infty} M+\lim_{n\to\infty} N=0,$$

that is,

$$\lim_{n\to\infty} p_b(x_0,x_n)+\lim_{n\to\infty} p_b(y_0,y_n)=0$$

implies

$$x_0=\lim_{n\to\infty} x_n \quad \text{and} \quad y_0=\lim_{n\to\infty} y_n. \tag{4.9}$$

Thus, the coupled fixed point is well-posed.
If we set $a_7=0$ in Theorem 4.3, then we obtain the following corollary, corresponding to Theorem 3.3 of [20].

Corollary 4.4 Let (X,p_b) be a complete partial b-metric space with coefficient $s\geq 1$ and let $f\colon X\times X\to X$ be a mapping. For any sequences $\{x_n\},\{y_n\}\subset X$ with $(x,y)\in C_0F(f)$, if

$$\lim_{n\to\infty} p_b(f(x_n,y_n),x_n)=0=\lim_{n\to\infty} p_b(f(y_n,x_n),y_n),$$

then the coupled fixed point problem of f is well-posed, and

$$p_b(x,x)=0 \quad \text{for some } x\in X.$$

5. Conclusion

In this paper, we establish some coupled fixed point results for contractive mappings in a partial b-metric space. Examples are provided to support the results and concepts presented herein. As an application of our results, we prove the well-posedness of coupled fixed point problems.

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References

[1] S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.*, 3:133–181, 1922.

[2] M. Raji and M. A. Ibrahim. Coincidence and common fixed points for F-Contractive mappings. *Ann. Math. Phys.*, 6(2):196–212, 2023.

[3] M. Raji and M. A. Ibrahim. Results in cone metric spaces and related fixed point theorems for contractive type mappings with application, Qeios, 2024, 17 pages.

[4] M. Raji and M. A. Ibrahim. Fixed point theorems for modified F-weak contractions via -admissible mapping with application to periodic points. *Annals of Mathematics and Computer Science*, 20:82–97, 2024.

[5] M. Raji. Generalized α-ψ Contractive Type Mappings and Related Coincidence Fixed Point The-orems with Applications. *The Journal of Analysis*, 31(2):1241–1256, 2023.

[6] M. Raji, A. K. Rajpoot, A. Hussain, L. Rathour, L. N. Mishra, and V. N. Mishra. Results in Fixed Point Theorems for Relational-Theoretic Contraction Mappings in Metric Spaces. *Tuijin Jishu / Journal of Propulsion Technology*, 45(1):4356–4368, 2024.

[7] M. Raji, A. K. Rajpoot, W. F. Al-omeri, L. Rathour, L. N. Mishra, and V. N. Mishra. Generalized α-ψ Contractive Type Mappings and Related Fixed Point Theorems with Applications. *Tuijin Jishu / Journal of Propulsion Technology*, 45(10):5235–5246, 2024.

[8] S. G. Matthews. Partial metric topology. In *Proceedings of the 8^{th} Summer Conference on Topology and its Applications*, volume 728, pages 183–197. Annals of the New York Academy of Sciences, 1994.

[9] M. Turinici. Fixed points for monotone iteratively local contractions. *Demonstr. Math.*, 19:171–180, 1986.

[10] A. C. M. Ran and M. C. B. Reurings. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Amer. Math. Soc.*, 132(5):1435–1443, 2004.

[11] J. J. Nieto and R. Rodriguez-Lopez. Contractive mapping theorems in partially ordered space and applications to ordinary differential equations. *Order*, 22(3):223–239, 2005.

[12] T. Gnana Bhaskar and V. Lakshmikantham. Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Analysis: TMA*, 65(7):1379–1393, 2006.

[13] S. Czerwik. Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica Univer-sitatis Ostraviensis*, 1:5–11, 1993.

[14] M. Boriceanu. Strict fixed point theorems for multivalued operators in b-metric spaces. *Int. J. Mod. Math.*, pages 285–301, 2009.

[15] H. Aydi. Some coupled fixed point results on partial metric spaces. *International J. Math. Math. Sci.*, page 11, 2011. Article ID 647091.

[16] S. Shukla. Partial b-metric spaces and fixed point theorems. 2013. doi:10.1007/s00009-013-0327-4.

[17] G. S. Saluja. Coupled fixed point results for contractive type conditions in partial b-metric spaces. *Anal. Math. Com. Sci.*, 14:2789–7206, 2023.

[18] M. Raji and M. A. Ibrahim. Fixed point theorems for fuzzy contractions mappings in dislocated b-metric spaces with applications. *Annals of Mathematics and Computer Science*, 21:1–13, 2024.

[19] H. K. Nashine and W. Shatanawi. Coupled common fixed point theorems for pair of commuting in partially ordered complete metric spaces. *Comput. Math. Appl.*, 62:1984–1993, 2011.

[20] M. Raji, L. Rathour, L. N. Mishra, and V. N. Mishra. Generalized Rational Type Contraction and Fixed Point Theorems in Partially Ordered Metric Spaces. *J. Adv. Appl. Comput. Math.*, 10:153–162, 2023.

[21] N. V. Luong and N. X. Thuan. Coupled fixed points in partially ordered metric spaces and application. *Nonlinear Anal.*, 74:983–992, 2011.

[22] H. Aydi. Some coupled fixed point results on partial metric spaces. *International J. Math. Math. Sci.*, 2011. Article ID 647091, 11 pages.

[23] S. Reich and A. J. Zaslavski. Well posedness of fixed point problem. *Far East J. Math.*, pages 393–401, 2001. Special Volume Part III.